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INEQUALITIES FOR THE DERIVATIVES OF POLYNOMIALS

R. P. BOAS, JR., Northwestern University

Some years after the chemist Mendeleev invented the periodic table of the elements he made a study of the specific gravity of a solution as a function of the percentage of the dissolved substance [16]. This function is of some practical importance: for example, it is used in testing beer and wine for alcoholic content, and in testing the cooling system of an automobile for concentration of anti-freeze; but present-day physical chemists do not seem to find it as interesting as Mendeleev did. Nevertheless, Mendeleev's study led to mathematical problems of great interest, some of which are still inspiring research today.

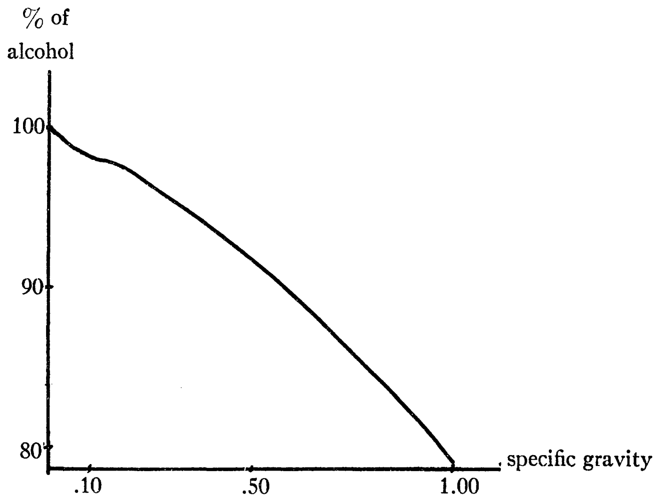

FIG. 1

An example of the kind of curve that Mendeleev obtained is shown in Figure 1 (alcohol in water, percentage by weight). He noticed that the curves could be closely approximated by successions of quadratic arcs, and he wanted to know whether the corners where the arcs joined were really there, or just caused by errors of measurement. In mathematical terms, this amounts to considering a quadratic polynomial $P(x)=px^2+qx+r$ on an interval $[a, b]$, with $\max P(x)-\min P(x)=L$, and asking how large $P'(x)$ can be on $[a, b]$. For, if the slope of one arc exceeds the largest possible slope for an adjacent arc, it follows that these arcs must come from different quadratic functions. We can reduce the problem to a simpler one by changing the horizontal scale and shifting the coordinate axes until the interval $[a, b]$ becomes $[-1, 1]$, and then changing the vertical scale and shifting the axes until we have $|P(x)|\leqq 1$. Our question then becomes, if $P(x)$ is a quadratic function and $|P(x)|\leqq 1$ on $[-1, 1]$, how large can $|P'(x)|$ be on $[-1, 1]$? The answer that Mendeleev found is that $|P'(x)|\leqq 4$; and this is the most that can be said, since when $P(x)=1-2x^2$ we have $|P(x)|\leqq 1$ and $|P'(\pm 1)|=4$. By using this result, Mendeleev convinced himself that the corners in his curves were genuine; and he was presumably right, since his measurements were quite accurate (they agree with modern tables to three or more significant figures).

One can readily imagine that a chemist who discovers such a pretty mathematical result would tell a mathematician about it; and in fact Mendeleev told it to A. A. Markov, who naturally investigated the corresponding problem for polynomials of degree n [13]. In particular, he proved what has come to be known as Markov's Theorem:

If $P(x)$ is a real polynomial of degree n, and $|P(x)| \leqq 1$ on $[-1, 1]$ then $|P'(x)| \leqq n^2$ on $[-1, 1]$, with equality attainable only at ± 1 and only when $P(x) = \pm T_n(x)$, where $T_n(x)$ (the so-called Chebyshev polynomial) is $\cos n \cos^{-1}x$ *(which actually is a polynomial, since* $\cos n\theta$ *is a polynomial in* $\cos \theta$*).*

Clearly we can also assert that if $|P(x)| \leqq L$ on $[-1, 1]$ then $|P'(x)| \leqq Ln^2$. Having now found an upper bound for $|P'(x)|$, it would be natural to go on and ask for an upper bound for $|P^{(k)}(x)|$ (where $k \leqq n$). Iterating Markov's theorem yields $|P^{(k)}(x)| \leqq n^{2k}L$ if $|P(x)| \leqq L$. However, this inequality is not sharp; the best possible inequality was found by Markov's brother, V. A. Markov, who proved that $|P^{(k)}(x)| \leqq T_n^{(2k)}(1)$ when $|P(x)| \leqq 1$; here T_n is again the Chebyshev polynomial. Explicitly,

$$|P^{(k)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdot 5\cdots(2k-1)}.$$

Later on we shall give a fairly simple proof of the inequality for $P'(x)$, but the inequality for $P^{(k)}(x)$ is considerably harder, except for $k=n$ (see, for example, [6], [21]; for $k=n$, Lemma 6, below).

The next similar question about polynomials was not asked for about 20 years, when S. Bernstein wanted, for applications in the theory of approximation of functions by polynomials, the analogue of Markov's theorem for the unit disk in the complex plane instead of for the interval $[-1, 1]$. He asked, if $P(z)$ is a polynomial of degree n and $|P(z)| \leqq 1$ for $|z| \leqq 1$, how large can $|P'(z)|$ be for $|z| \leqq 1$? The answer is that $|P'(z)| \leqq n$, with equality attained for $P(z) = z^n$. Bernstein's problem can be stated in a different way which suggests many interesting generalizations and has a number of applications. Since a polynomial $P(z)$ is an analytic function, it attains its maximum absolute value for $|z| \leqq 1$ on the circumference $|z| = 1$; so does its derivative. Hence if we want $\max|P'(z)|$ for $|z| \leqq 1$ given that $|P(z)| \leqq 1$ for $|z| \leqq 1$, it is enough to consider only values of z with $|z| = 1$, that is, $z = e^{i\theta}$ with $0 \leqq \theta < 2\pi$. Now $P(e^{i\theta})$ can be written as a linear combination of sines and cosines,

$$S(\theta) = \sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta);$$

such an expression is called a trigonometric sum of degree n (or a trigonometric polynomial). Hence Bernstein's theorem can be restated as follows:

If $S(\theta)$ is a trigonometric sum of degree n (possibly with complex coefficients) and $|S(\theta)| \leqq 1$, then $|S'(\theta)| \leqq n$, with equality attained when $S(\theta) = \sin n(\theta - \theta_0)$.

As Bernstein observed, if $P(x)$ is our original polynomial on $(-1, 1)$, $P(\cos\theta)$ is a trigonometric sum of degree n, and so $|P'(\cos\theta)\sin\theta| \leqq n$ by Bernstein's theorem, which is to say that $|P'(x)| \leqq n(1-x^2)^{-1/2}$ for $|x| \leqq 1$. This gives an estimate for $|P'(x)|$ that is much better than Markov's when x is not near ± 1, but it does not yield Markov's theorem directly since it tells us nothing about $|P'(x)|$ when x is near ± 1. It is rather remarkable that Markov's theorem can nevertheless be deduced from Bernstein's theorem.

There are many proofs of Bernstein's and Markov's theorems. Those given here are interesting because they demand very little machinery, and illustrate how unexpected results can sometimes be obtained from very simple considerations. We begin by stating some almost obvious results as lemmas.

LEMMA 1. *The polynomial $T_n(x)$ takes the values $+1$ and -1 a total of $n+1$ times in the interval $[-1, 1]$, with alternating signs.*

In fact, $T_n(x) = \pm 1$ whenever $n \cos^{-1}x = k\pi$, $0 \leqq k \leqq n$.

LEMMA 2. *If two polynomials of degree at most n have the same values at $n+1$ points, they are the same polynomial.*

LEMMA 3. *If two trigonometric sums of degree n have the same values at $2n+2$ points in $0 \leqq \theta \leqq 2\pi$ (counting both ends), they are the same.*

This follows from Lemma 2. A trigonometric sum $S(\theta)$ of degree n can be written

$$\sum_{k=-n}^{n} c_k e^{ik\theta} = e^{-in\theta} \sum_{k=0}^{2n} c_{k-n} e^{ik\theta},$$

i.e., as $e^{-in\theta}$ times a polynomial of degree $2n$ in $e^{i\theta}$, and accordingly has (if not identically 0) at most $2n+1$ zeros in $[0, 2\pi]$ (allowing for the fact that a zero at 0 is repeated at 2π).

LEMMA 4. *Suppose that ϕ and f are real-valued continuous functions on $-1 \leqq x \leqq 1$, that ϕ takes the values ± 1 at $k+1$ points x_j with alternating signs, and that $|f(x_j)| < 1$ for all the points x_j. Then there are k points where $f(x) = \phi(x)$.*

In geometrical terms, if the graph of ϕ has k arcs connecting the line $y=1$ with $y=-1$, and the graph of f is between these lines at the points where the graph of ϕ meets them, the graphs of f and ϕ have k intersections. Indeed, if (for example) $\phi(x_j) = -1$ and $\phi(x_{j+1}) = +1$, we have $\phi(x) - f(x)$ negative at x_j and positive at x_{j+1}, and so zero somewhere in between.

LEMMA 5. *Under the hypotheses of Lemma 4, if the graph of f crosses the graph of ϕ from below to above on an arc that rises from -1 to $+1$, then the graphs of f and ϕ cross at at least $k+2$ points.*

The situation is illustrated in Figure 2. Let the arc specified in the lemma connect $(a, -1)$ with $(b, 1)$, $b > a$, and let the crossing occur at $x = c$. Since the graph of f is above that of ϕ at $x = a$, below it somewhere between $x = a$ and $x = c$, above it between c and b, and below it again at $x = b$, there are at least 3

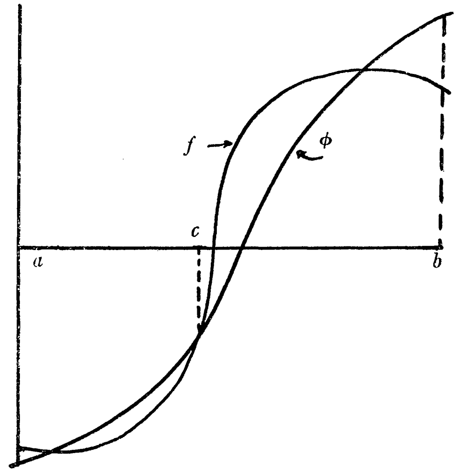

FIG. 2

crossings on this arc, and $k-1$ on the remaining arcs of the graph of ϕ, so $k+2$ in all.

We begin by using Lemma 5 to prove Bernstein's theorem (for a similar proof, cf. [11]). We begin with a real trigonometric sum (i.e., one with real coefficients); the extension to sums with complex coefficients depends on a trick, which we give at the end of the proof .

We are, then, given a real trigonometric sum $S(\theta)$ with $|S(\theta)| \leqq 1$; we shall suppose that $S'(\theta_0) > n$ for some θ_0, and obtain a contradiction. Since we can, if necessary, replace $S(\theta)$ by $\lambda S(\theta)$ with λ slightly less than 1, and still have $\lambda S'(\theta_0) > n$, we may assume $|S(\theta)| < 1$. Take the graph of $\sin n\theta$ and slide it horizontally until one of its arcs with positive slope meets the graph of $S(\theta)$ at θ_0. At this point $S(\theta)$ has a larger slope than the shifted sine curve (whose slope is at most n). By Lemma 5, the shifted sine curve and the graph of $S(\theta)$ meet at $2n+2$ points between 0 and 2π; by Lemma 3, $S(\theta)$ is itself a shifted sine curve, a contradiction since $|S(\theta)| < 1$.

Now we consider trigonometric sums with complex coefficients. It is easy to show that for each θ_0 there is a trigonometric sum $S_0(\theta)$ of degree n for which $|S_0(\theta)| \leqq 1$ and $|S_0'(\theta)|$ has the largest possible value—this depends on the fact that $S(\theta)$ has only a finite number of coefficients, whose possible values are in a bounded set since $|S(\theta)| \leqq 1$; we then appeal to the principle that a real-valued continuous function (namely $|S_0'(\theta_0)|$) defined over a compact set (the $(2n+1)$-dimensional set of possible coefficients) attains its maximum. Since $S(\theta-\theta_0)$ is a trigonometric sum if $S(\theta)$ is, there is no loss of generality in assuming that $\theta_0=0$. Take, then, an $S_0(\theta)$ such that $|S_0'(0)|$ is as large as possible; we have to show that $|S_0'(0)| \leqq n$. We can choose a real λ such that $e^{i\lambda}S_0'(0) > 0$; having done this, consider the real trigonometric sum $\mathrm{Re}(e^{i\lambda}S_0(\theta))$, whose absolute value does not exceed 1. Its derivative has absolute value not exceeding n, by what we already know; in particular, this is true at $\theta=0$, so that $0 < e^{i\lambda}S_0'(0) \leqq n$. Hence $|S_0'(0)| \leqq n$, as asserted.

Observe that although Markov's theorem cannot be iterated, as we saw, Bernstein's theorem can; in other words, $|S''(\theta)| \leqq n^2$, and so on for higher derivatives. The bounds so obtained are best possible, as is shown by $S(\theta) = \sin n\theta$.

We now turn to the proof of Markov's theorem. It is conceivable that it could be proved directly that $|P'(x)|$ attains its maximum at $x = \pm 1$. If we could do this, Markov's theorem would follow at once from Bernstein's, since $P(\cos\theta)$ is a trigonometric sum $S(\theta)$ and we have

$$|S''(\theta)| = |P''(\cos\theta)\sin^2\theta - P'(\cos\theta)\cos\theta| \leqq n^2;$$

putting $\theta = 0$ and $\theta = \pi$, we would obtain $|P'(\pm 1)| \leqq n^2$.

Unfortunately we do not yet know that $|P'(x)|$ attains its maximum at ± 1. However, since $P(\cos\theta)$ is a trigonometric sum, we do have $|P'(x)| \leqq n(1-x^2)^{-1/2}$, and if $|x| \leqq \cos(\frac{1}{2}\pi/n)$ this gives us

$$|P'(x)| \leqq n\{1 - \cos^2(\tfrac{1}{2}\pi/n)\}^{-1/2} = n \csc(\tfrac{1}{2}\pi/n) \leqq n^2$$

because $|\sin n\theta| \leqq n|\sin\theta|$ for all real θ (here $\theta = \frac{1}{2}\pi/n$). Note that if $|P(x)| < 1$ for $-1 \leqq x \leqq 1$ we have strict inequality here. Thus Markov's theorem is established except for $\cos(\frac{1}{2}\pi/n) < x < 1$ (and the symmetric interval).

To handle the excluded intervals we need another auxiliary result, which is of independent interest.

LEMMA 6 (Chebyshev's theorem). *Let $P(x)$ be a real polynomial of degree n, such that $|P(x)| \leqq 1$ on $[-1, 1]$; then the leading coefficient of $P(x)$ has absolute value at most 2^{n-1} (which is the leading coefficient of $T_n(x) = \cos n \cos^{-1} x$).*

Suppose, in fact, that $P(x)$ has a leading coefficient larger than 2^{n-1}; this is still true for $\lambda P(x)$ with some $\lambda < 1$, so that we may assume $|P(x)| < 1$ on $[-1, 1]$. The polynomial $T_n(x)$ takes the values ± 1 with alternating signs at $n+1$ points, so Lemma 4 applies with $\phi = T_n$, $f = P$: we have $P(x) = T_n(x)$ at n distinct values of x on $[-1, 1]$. In addition $P(1) < T_n(1) = 1$ but $P(x) > T_n(x)$ for large positive x, since $P(x)$ has a larger leading coefficient than $T_n(x)$. Hence there is an $(n+1)$th point where $P(x) = T_n(x)$; consequently $P(x) \equiv T_n(x)$, a contradiction. (For another elementary proof, see [19].)

We have actually proved a stronger result: we only need to assume $|P(x)| \leqq 1$ at the points where $|T_n(x)| = 1$, and we can add to the conclusion the statement that $|P(x)| \leqq |T_n(x)|$ for $|x| > 1$.

We now return to the problem of establishing Markov's theorem for $\cos(\frac{1}{2}\pi/n) < x < 1$. We shall actually prove somewhat more, namely that $|P'(x)| \leqq T_n'(x)$ for $x > \cos(\frac{1}{2}\pi/n)$ (including $x > 1$). Suppose, in fact, that $P'(x_0) > T_n'(x_0)$ for some $x_0 > \cos(\frac{1}{2}\pi/n)$; we seek to obtain a contradiction. As before, we may assume that $|P(x)| < 1$ for $|x| \leqq 1$. We may also suppose $n > 1$, since the theorem is trivial when $n = 1$. There are $n-1$ arcs of the graph of $T_n(x)$, each connecting $y = -1$ with $y = 1$, to the left of the point $\cos(\frac{1}{2}\pi/n)$, and x_0 is on the nth one or on its continuation to $x > 1$. The graph of $P(x)$ crosses each of the $n-1$ arcs, so that $P(x) = T_n(x)$ at (at least) $n-1$ points to the left of $\cos(\frac{1}{2}\pi/n)$, and therefore $P'(x) = T_n'(x)$ at $n-2$ points to the left of $\cos(\frac{1}{2}\pi/n)$. We have already seen that $|P'(\cos(\frac{1}{2}\pi/n))| < T_n'(\cos(\frac{1}{2}\pi/n))$; we have assumed $P'(x_0) > T_n'(x_0)$; and finally, by Lemma 6, we have $P'(x) < T'(x)$ for very large x. Therefore $P' = T_n'$ at (at least) two points to the right of $\cos(\frac{1}{2}\pi/n)$, making n points in all. But $P' - T_n'$ is of degree $n-1$ at most, and so vanishes identically. Hence P and T_n differ at most by a constant, which must be zero since they coincide at $n-1$ points. Since $|P(x)| < 1$, we have a contradiction. (Cf. [**12**].)

When $x > 1$, we get, in particular, $|P'(x)| < n^2(x + (x^2-1)^{1/2})^{n-1}$, which was known to Mendeleev when $n = 2$.

Markov's and Bernstein's theorems have been generalized in a great variety of ways. Unfortunately, most of the generalizations cannot be proved by the elementary methods we have just used, so we shall only be able to state some typical results without proof. Each of the three theorems (Markov's, Bernstein's, and Chebyshev's) is a special case of a general problem: we have a class of functions (polynomials of degree at most n, or trigonometric sums of degree at most n) each of absolute value not exceeding 1, and to each function we attach a number (the maximum of the derivative in the Bernstein and Markov cases, the leading coefficient in Chebyshev's case); in technical language, we have a functional defined over our class of functions. We then want to maximize the functional over the given class.

Recall that Bernstein's theorem, when applied to polynomials P of absolute value at most 1 on $[-1, 1]$, gave the result that $|P(x)| \leqq n(1-x^2)^{-1/2}$. We can put this in our current framework by picking a value of x and taking our functional to be $|P'(x)|$. In other words, we ask, how large can $|P'(x)|$ be, for a given x, when $|P(x)| \leqq 1$ on $[-1, 1]$? Call this maximum $M_n(x)$. Then Markov's theorem says that $M_n(x) \leqq n^2$ for $-1 \leqq x \leqq 1$, and Bernstein's theorem says that $M_n(x) \leqq n(1-x^2)^{-1/2}$ for $-1 < x < 1$. The problem of finding $M_n(x)$ exactly was attacked by Markov himself, and solved explicitly for $n = 2$ and $n = 3$. Since it is easy to see that $M_n(-x) = M_n(x)$, it is enough to find $M_n(x)$ for $x \geqq 0$. Although we already know that $M_n(1) = T_n'(1)$, it is clear that $M_n(x)$ cannot always be $|T_n'(x)|$, since $T_n'(x)$ is sometimes zero; we know, however, from our proof of Markov's theorem that $M_n(x) = T_n'(x)$ for $x > \cos(\frac{1}{2}\pi/n)$, and in particular for $x > 1$. Markov found that $M_2(x) = T_2'(x) = 4x$ for $(1/2) < x < 1$, but $M_2(x) = 1/(1-x)$ for $0 \leqq x < (1/2)$. It follows that Bernstein's estimate for $M_2(x)$ is exact for just one x in $[0, 1]$, namely $x = 2^{-1/2}$ (see Figure 3). The function $M_3(x)$ is much more complicated (see the appendix to this paper, and Figure 4). Calculation of $M_n(x)$ for larger values of n had to wait until quite recently, when, as a culmination of some 30 years of work, E. V. Voronovskaja [**25**] produced a technique that not only lets one calculate $M_n(x)$ for any n (graphs for $n = 4$ and 5 are given in [**26**] and [**25**]), but makes it possible to solve many other problems. For example, Gusev [**8**] finds the function corresponding to $M_n(x)$ for the functional $P^{(k)}(x)$ $(1 < k < n)$. If $P(x)$ is a polynomial with real coefficients and $|P(x)| \leqq 1$ on $[-1, 1]$ then Voronovskaja and Zinger determined $\max|\operatorname{Re} P(z)|$ and $\max|\operatorname{Im} P(z)|$ for a given complex z [**27**], and Zinger [**28**], [**29**], determined the corresponding maxima for the derivatives of P.

The maximizing functions for a given functional are usually not Chebyshev polynomials; and when they are not, elementary methods do not usually work.

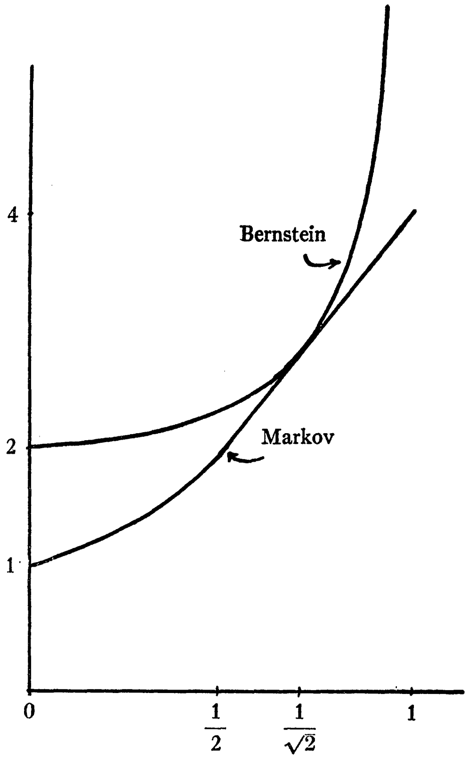

FIG. 3

FIG. 4

In fact, every polynomial is the maximizing polynomial for some functional [25], [18].

The situation for functionals over trigonometric sums is similar.

However, merely considering different functionals does not exhaust the possibilities of generalization. In the first place one can try to maximize a functional under some side-condition, for example that all the functions taken into consideration vanish at a given point. The simplest such problem is perhaps Schur's problem of maximizing $P_n(x)$ (for a given x) when P_n is a polynomial of degree n such that $|P_n(x)| \leqq 1$ on $[-1, 1]$ and $P_n(0)=0$. This is again an elementary problem, and the answer is that $|P_n(x)| \leqq |\sin m \sin^{-1} x|$ for $|x| < \sin \frac{1}{2}\pi/m$, where $m=n$ or $n-1$ according as n is odd or even; otherwise no more than $|P_n(x)| \leqq 1$ can be asserted. Hyltén-Cavallius [10] solved the more general problem when the value of $P_n(z_0)$ is assigned for an arbitrary (real or complex) z_0. The presence of side-conditions does not complicate the problems appreciably, but neither does it simplify them.

In the Markov theorems we found a bound for a functional given $\max|P_n(x)|$ on $[-1, 1]$; in Bernstein's theorem, the maximum was taken over the unit disk. We can ask the same questions given a bound for $|P_n(z)|$ on any specified subset of the plane, or for $|S_n(\theta)|$ on a subset of a period (or indeed on a subset of the plane). (Cf. [3].)

An essential difference, which is reflected in a difference in method, arises if one restricts the class of functions in a different way. For example, consider polynomials of degree n all of whose roots are real and outside $[-1, 1]$; then if $|P(x)| \leqq 1$ on $[-1, 1]$ it follows that $|P'(x)| < \frac{1}{2}en$ (and the constant is best possible) [7].

There are also many new problems when the polynomials or trigonometric sums are restricted to be nonnegative. For example, if $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a trigonometric sum of degree n then $|c_k| \leqq \frac{1}{2}\max|S_n(\theta)|$ for $k > \frac{1}{2}n$ (the inequality with 1 instead of $\frac{1}{2}$ is trivial); many similar but more complicated inequalities are known. A related condition to impose on a trigonometric polynomial is that it is a partial sum of the Fourier series of a nonnegative function (this does not force it to be nonnegative itself). For example, if the function is even as well as nonnegative, we have $|c_k|^2 \leqq \frac{1}{2}(1+c_{2k})$, an inequality that seems to have been discovered by crystallographers before it was noticed by mathematicians (cf. [9]).

On the other hand, one can also generalize the problem by enlarging the class of functions under consideration. In general, the maximum of a given functional over a larger class of functions can of course be expected to be larger than over the smaller class. In some cases, however, it turns out to be the same. A trigonometric sum $\sum_{k=-n}^{n} c_k e^{ikx}$ is a special case of a finite Fourier-Stieltjes transform $\int_{-n}^{n} e^{ixt} d\alpha(t)$; another special case is an integral of the form $\int_{-n}^{n} e^{ixt} g(t)dt$. This occurs in communication theory under the name of a band-limited signal (see, e.g., [17]), and it also occurs in the theory of optical instruments, antennas, and other kinds of electromagnetic apparatus [4]. It turns out that Bernstein's theorem on trigonometric sums extends to functions of this form (and even to a larger class of functions) without change (but naturally with a quite different proof), and has physical significance. There are also inequalities for nonnegative finite Fourier transforms corresponding to those for nonnegative trigonometric sums.

Still another possibility is to recognize that $\max|S_n(\theta)|$, for example, is just the norm usually used for the space of continuous functions, of which the trigonometric sums form a subset. They are also a subset of the space of functions of integrable pth power, in which the norm is

$$\left\{\int_{-\pi}^{\pi} |f(\theta)|^p d\theta\right\}^{1/p}.$$

The exact analogue of Bernstein's theorem (due to Zygmund) holds here ([30], vol. 2, p. 11; see also [23]):

$$\left\{\int_{-\pi}^{\pi} |S_n'(\theta)|^p d\theta\right\}^{1/p} \leqq n\left\{\int_{-\pi}^{\pi} |S_n(\theta)|^p d\theta\right\}^{1/p}, \qquad p \geqq 1;$$

Bernstein's theorem is the limiting case $p \to \infty$.

Again, one can extend all the problems we have considered by asking for generalizations to higher dimensions. Here the difficulty is often not so much that of proving the theorems as of discovering what would be interesting to prove. The most interesting results are those that do not have an analogue in one dimension. One illustration is as follows. A polynomial $P(z)$ in the complex variable $z=x+iy$ can be written as $R(x, y)+iS(x, y)$, where $R(x, y)$ is a harmonic polynomial (i.e., a polynomial solution of Laplace's equation). Since

$$P'(z) = \frac{\partial P}{\partial r} = \frac{\partial R}{\partial r} + i\frac{\partial S}{\partial r} = \frac{\partial R}{\partial r} + \frac{i}{r}\,\frac{\partial R}{\partial \theta},$$

the real and imaginary parts of P' are the (polar) components of the vector grad R. Bernstein's theorem says that if $|P(z)| \leqq 1$ for $|z| \leqq 1$ then $|P'(z)| = |\text{grad } R| \leqq n$. Szegö (cf. [22], [24]) proved more: if we assume only that $|R| \leqq 1$ then $|\text{grad } R| \leqq n$; interpreting this in rectangular coordinates, we have

$$\left(\frac{\partial R}{\partial x}\right)^2 + \left(\frac{\partial R}{\partial y}\right)^2 \leqq n^2.$$

It would be interesting to have an elementary proof of this along the preceding lines. Szegö extended the theorem to three dimensions, where the bound for grad P is more complicated; in more than three dimensions the problem seems to be unsolved.

So much has been written on Bernstein's and Markov's theorems and their generalizations that it is hardly possible to give a complete bibliography. The following list of references includes, besides items specifically cited, a number of books and papers that contain additional results or interesting methods of proof.

Appendix

Here is the explicit form of Markov's function $M_3(x)$ for $0 \leqq x \leqq 1$; its graph is sketched in Figure 4.

x	$M_3(x)$
$0 \leqq x \leqq \frac{\sqrt{7}-2}{6} = 0.108$	$3(1-4x^2)$
$\frac{\sqrt{7}-2}{6} \leqq x \leqq \frac{2\sqrt{7}-1}{9} = 0.47$	$\frac{7\sqrt{7}+10}{9(x+1)}$
$\frac{2\sqrt{7}-1}{6} \leqq x \leqq \frac{1+2\sqrt{7}}{9} = 0.70$	$\frac{16x^3}{(1-9x^2)(1-x^2)}$
$\frac{1+2\sqrt{7}}{9} \leqq x \leqq \frac{\sqrt{7}+2}{6} = 0.774$	$\frac{7\sqrt{7}-10}{9(1-x)}$
$\frac{\sqrt{7}+2}{6} \leqq x \leqq 1$	$3(4x^2-1)$

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SEPARABLE FUNCTIONS AND THE GENERALIZATION OF MATRICIAL STRUCTURE

MILTON ROSENBERG, Kansas University

1. Introduction. Our main concern in this paper shall be the *separability* of a function in accordance with the following definition. The reader is assumed to have some background in linear algebra.

DEFINITION 1. *Let B and T be sets and let $f(b, t)$ be a function from $B \times T$ into a field F. We say that f is separable if it can be expressed in the form*

$$\sum_{i=1}^{n} g_i(b)h_i(t) \tag{I}$$

where the g_i's and h_i's are functions defined on B and T respectively, with their values in F.

We may call (I) a *dyadic representation.* A dyadic representation of a $k \times m$ matrix A is given by $A = \sum_{i=1}^{n} g_i \cdot h_i'$, where the g_i's and h_i's are k-tuple and m-tuple column vectors, respectively ("$'$" means "transpose"). The function $f(b, t)$ may be viewed as a matrix where the b's parametrize the rows and the t's, the columns. Thus it is natural that we next define its rank. From Lemma 1, (Section 4), it will follow that this is an extension of the usual definition for the rank of a matrix.

DEFINITION 2. (a) *f is separable of rank zero if $f(b, t) \equiv 0$. (Notation, rank(f)* $=0$).

(b) *f, $\not\equiv 0$, is separable of rank n if f is separable and if n is the smallest integer such that f can be represented in the form* (I). (*Notation, rank$(f) = n$*).

(c) *f is called inseparable and is said to have rank infinity* (*rank$(f) = \infty$*) *if f is not separable.*

Remark. In Definitions 1 and 2, we logically should have used the terms "F-separable" and "F-separable of rank n," with the corresponding notation "$\text{rank}_F(f) = n$." However, it will be shown in Section 4, Theorem 1, that if we replace F by any other field K containing the range of f, then rank $_K(f) = \text{rank}_F$ (f).

2. Our results. In Section 3 (Contexts) we shall see that the topological (i.e., continuity) and the analytical (i.e., differentiability) properties of separable functions are often important. However, this paper solely concerns itself with the algebraic nature of separability. First we summarize basic characterizations of separability (Section 4, Lemma 1), prove Theorem 1 justifying the remark of Section 1, and then give some examples. We investigate the ranks of sums and products of separable functions in Lemma 4. Next we develop a general method for proving inseparability (Section 5, Theorem 3). In Section 6 we completely solve the problem of the separability of quotients of polynomial functions (Theorem 4). In Section 7 we mention the problem of pure separability and pure inseparability. Finally, we conclude the paper (Section 8).

3. Contexts. Although it is only in integral equations that separable functions are distinguished by a special name, "degenerate kernels" (cf. [2]), they occur in:

(1) *Ordinary differential equations.* The equation $db/dt = g(b)h(t)$ is commonly called "separable" (cf. [7] for its history). In [14, p. 8] the solution of $dy/dx = \sin(xy)$ with initial conditions $(x_0, y_0) = (0, 0)$ is approximated by the solution of $dy/dx = xy$; by a general theorem, it is seen that the error is less than $(.6/192)|x|$ for $|x| < \frac{1}{2}$.

(2) *Partial differential equations.* Solutions are often obtained by the "method of separation of variables" (cf. [5], [2] vol. 2). For example [2, vol. 2, 18–20], to solve $(\partial u/\partial x)^2 + (\partial u/\partial y)^2 = 1$, try $u = \varphi(x) + \psi(y)$. To solve the heat equation $u_{xx} = u_y$, set $u = \varphi(x) \cdot \psi(y)$, obtaining the solutions $(\cos \nu x) \cdot \exp(-\nu^2 y)$; differentiation, summation, or integration over the parameter ν again gives us solutions, e.g.,

$$u = \int_{-\infty}^{\infty} e^{-v^2 y} \cos vx dv = \sqrt{(\pi/y)} e^{-x^2/4y}$$

is the "fundamental solution."

(3) *Integral equations.* In the theory of the Fredholm integral equation, $\varphi(s) - \lambda \int_a^b K(s, t)\varphi(t)dt = f(s)$ (cf. [2] vol. I, p. 115 and [14] p. 119) where it is desired to obtain a solution $\varphi(t)$, general results are obtained by approximating the continuous kernel $K(s, t)$ by continuous degenerate kernels and taking limits.

(4) *Statistics* (Multiple regression, (cf. [10] p. 343)). This topic gave rise to the present paper. Assume we are observing the x-component of a motion given by a known separable time function, $x(b, t)$, in the form (I), where b is an unknown parameter. Assume we make our observations $s(b, t) = x(b, t) + e(t)$ ($e(t) =$ error) at a sequence of times $t_1, \cdots, t_N (N \geqq n)$. We have

$$s = \begin{bmatrix} s(b,t_1) \\ \vdots \\ s(b,t_N) \end{bmatrix} = \begin{bmatrix} h_1(t_1) & h_2(t_1) & \cdots & h_n(t_1) \\ \vdots & \vdots & & \vdots \\ h_1(t_N) & h_2(t_N) & \cdots & h_n(t_N) \end{bmatrix} \cdot \begin{bmatrix} g_1(b) \\ \vdots \\ g_n(b) \end{bmatrix} + \begin{bmatrix} e(t_1) \\ \vdots \\ e(t_N) \end{bmatrix};$$

i.e., $s = H \cdot g + e$. If the errors are uncorrelated with identical variance and if $\operatorname{rank}(H) = n$, then the best unbiased linear estimates of the $g_i(b)$'s and of their linear combinations exist and are given by

$$\hat{g} = (H' \cdot H)^{-1} H' \cdot s, \quad \hat{x}(b, t) = [h_i(t) \cdots h_n(t)] \cdot \hat{g}, \quad \text{etc.}$$

In general, estimation of the true value of the parameter b is still a problem, even when $x(b, t)$ is separable.

4. Generalities and examples. Parts (a), (e), and (f) of Lemma 1 tell us that either the column and row ranks of $f(b, t)$, viewed as a matrix, are both finite and equal to the rank of f, or that the row and column ranks are both infinite. Only part (b) seems to be clearly stated in the literature (cf. [2] vol. I, p. 114).

LEMMA 1. (Separability). *Let $f \neq 0$ be a function from $B \times T$ into a field F.*

Then the following conditions (a), (b), (c), (d), (e), *and* (f) *are equivalent*:

(a) *f is separable of rank n.*

(b) *There exist n linearly independent functions $g_i(b)$ and n linearly independent functions $h_i(t)$ such that f can be expressed in the dyadic form* (I).

(c) *The maximum rank of the (square or rectangular) matrices $[f(b_i, t_j)]$ for all possible finite choices of distinct b_i's and t_j's is equal to n.*

(d) *There exist $b_1, \cdots, b_n, t_1, \cdots, t_n$ such that the matrix $M=[f(b_i, t_j)]$ is nonsingular, and for any such $b_1, \cdots, b_n, t_1, \cdots, t_n$ the following formula holds*:

$$f(b, t) = [f(b, t_1) \cdots f(b, t_n)] \cdot M^{-1} \cdot \begin{bmatrix} f(b_1, t) \\ \vdots \\ f(b_n, t) \end{bmatrix}. \tag{II}$$

(e) *The vector space over F spanned by the functions $\{f(\cdot, t)\}_{t \in T}$ has dimension n.*

(f) *The vector space over F spanned by the functions $\{f(b, \cdot)\}_{b \in B}$ has dimension n.*

Proof. We show first that (a)$\Rightarrow$(b): f is separable of rank n means we can express f in the form (I). Suppose the g's are linearly dependent, say $g_1 = \sum_{j=2}^{n} a_j g_j$. Then

$$f(b, t) = \sum_{j=2}^{n} g_j(b) \{ h_j(t) + a_j h_1(t) \},$$

and hence f is separable of rank $n-1$ or less, a contradiction.

Next, (c)$\Rightarrow$(d): Let

$$Q = \left[\begin{array}{c|c} f(b_1, t) & f(b_1, t_1) \cdots f(b_1, t_n) \\ \vdots & \vdots \\ f(b_n, t) & f(b_n, t_1) \cdots f(b_n, t_n) \\ \hline f(b, t) & f(b, t_1) \cdots f(b_1 t_n) \end{array}\right] = \left[\begin{array}{c|c} L & M \\ \hline f(b, t) & N \end{array}\right]$$

where M has rank n. Then the last row of Q is linearly dependent on the other rows. Therefore, there exists a unique row vector $[a_1 \cdots a_n]$ such that $[f(b,t) | N] = [a_1 \cdots a_n] \cdot [L | M]$. Thus $N = [a_1 \cdots a_n] \cdot M$ and hence $[a_1 \cdots a_n] = N \cdot M^{-1}$. Therefore $f(b, t) = [a_1 \cdots a_n] \cdot L = N \cdot M^{-1} \cdot L$.

Next, (d)$\Rightarrow$(e) ($d \Rightarrow f$ is similar): We note by equation (II) that $\{f(\cdot, t_1), \cdots, f(\cdot, t_n)\}$ spans $\{f(\cdot, t)\}_{t \in T}$. The linear independence of the $f(\cdot, t_j)$ follows from the nonsingularity of the matrix $M = [f(b_i, t_j)]$, i.e., the linear independence of its column vectors.

Now, for use in the proof of "(e)$\Rightarrow$(a)", we prove

LEMMA (A). *If $f(b, t) = \sum_{i=1}^{n} g_i(b) h_i(t)$, then for all finite choices of b_j, t_k we have*

$$\max \mathit{rank}\, [f(b_j, t_k)] = \bar{n} \leqq n. \tag{1}$$

Proof of Lemma. (1) follows because $[f(b_j, t_k)] = [\sum_{i=1}^n g_i(t_j)h_i(b_k)] = G \cdot H$, where the ranks of the matrices G and H are at most n. (The rank of a product of matrices is less than or equal to the rank of each factor in the product (cf. [**6**]).

(e)$\Rightarrow$(a) (f$\Rightarrow$a is similar): Let $\{f(\cdot, t_1), \cdots, f(\cdot, t_n)\}$ span $\{f(\cdot, t)\}_{t\in T}$. Then for each $t\in T$ there exists a unique set of coefficients $h_1(t), \cdots, h_n(t)$ such that

$$f(\cdot, t) = \sum_{i=1}^{n} f(\cdot, t_i)h_i(t).$$

On defining $f(\cdot, t_i) = g_i(\cdot)$ we have $f(b, t) = \sum_{i=1}^n g_i(b)h_i(t)$. We now show that n is the smallest integer such that f may be written in the form (I). Assume there is a smaller integer $m < n$. Then, by Lemma(A), max rank $[f(b_j, t_k)] = \bar{n} \leqq m$. Since (c)$\Rightarrow$(d)$\Rightarrow$(e), we obtain $\bar{n}(<n)$ as the dimension of the space spanned by $\{f(\cdot, t)\}_{t\in T}$, a contradiction.

For proving (b)$\Rightarrow$(c) we shall use

LEMMA (B). *Let $\{g_i(b)\}$, $i=1, \cdots, n$, be a set of linearly independent functions over the field of scalars F. Then*

(i) *The space S spanned by the n-tuples $\{g_1(b), g_2(b), \cdots, g_n(b)\}$ for $b\in B$ has dimension n.*

(ii) *There exist $b_1, \cdots, b_n$ such that the matrix $[g_i(b_j)]$, $i, j=1, \cdots, n$, is nonsingular, i.e., has rank n.*

Proof. We shall prove (i) by contradiction. (For the fact that $\dim(S) = m \leqq n$, we refer to [**1**, pp. 166–168].) Now, suppose m is less than n. Let $b_1, \cdots, b_m$ parametrize a basis $\{g_1(b_j), \cdots, g_n(b_j)\}$, $j=1, \cdots, m$, of S. Then, by deletion of columns and relabeling we may form the matrix

$$Q = \left[\begin{array}{c|c} g_i(b_1) & g_1(b_1) \cdots g_m(b)_1 \\ \vdots & \vdots \\ g_i(b_m) & g_1(b_m) \cdots g_m(b_m) \\ \hline g_i(b) & g_1(b) \cdots g_m(b) \end{array}\right] = \left[\begin{array}{c|c} L & M \\ \hline g_i(b) & N \end{array}\right]$$

where M is nonsingular. As in the proof that (c)$\Rightarrow$(d), we deduce

$$g_i(b) = [g_1(b) \cdots g_m(b)] \cdot M^{-1} \cdot L, \qquad i = m+1, \cdots, n,$$

contradicting the linear independence of the g's. (ii) follows immediately from (i).

$$\text{(b)} \Rightarrow \text{(c): From Lemma (A) we have max rank } [f(b_j, t_k)] \leqq n.$$

Let $b_1, \cdots, b_n$ be chosen as in Lemma (B) (ii); similarly let $t_1, \cdots, t_n$ be chosen. Then $[f(b_j, t_k)] = [\sum_{i=1}^n g_i(b_j)h_i(t_k)]$, $j, k=1, \cdots, n$, is the product of two nonsingular matrices and thus has rank n; i.e., max rank $[f(b_j, t_k)] = n$.

Thus we have shown

$$\begin{array}{ccccccccc} (\mathrm{a}) & \Rightarrow & (\mathrm{b}) & \Rightarrow & (\mathrm{c}) & \Rightarrow & (\mathrm{d}) & \Rightarrow & (\mathrm{e}) \\ & & & & & & \Downarrow & & \Downarrow \\ & & & & & & (\mathrm{f}) & \Rightarrow & (\mathrm{a}) \end{array}$$

We mention that in the preceding proof, (c)⇒(d) is an immediate consequence of the "bordering method" of inverting a matrix [**3**, p. 105] and that in essence its proof is also contained in [**9**, p. 11] and [**12**, p. 762].

For completeness, we state the proof of (a)⇒(b) as

LEMMA 2. f, $\neq 0$, *is separable of order n implies that for any representation of* f *in the form* (I) *with integer n, the g's and h*'s, *respectively, are linearly independent.*

Also, we may strengthen Lemma(B) by the following:

COROLLARY (to Lemma 1). *The dimension of the linear space spanned by the functions* $\{f_i(a)\}$, $i=1, \cdots, n$, *a in A, and the dimension of the linear space spanned by the n-tuples* $\{f_1(a), f_2(a), \cdots, f_n(a)\}$ *for* $a \in A$ *are equal.*

Proof. Follows immediately from Lemma (1) (e) and (f), on noting that $f(i, a)$ defined equal to $f_i(a)$ is separable.

We now justify the remark of Section 1. The smallest field containing the range of $f(b, t)$ is the (set)-intersection of all the fields which contain the range of $f(b, t)$.

THEOREM 1. *Let F be the smallest field containing the range of f and let K be any other field such that* $\text{range}(f) \subseteq K$. *Then* $\text{rank}_F(f) = n \Leftrightarrow \text{rank}_K(f) = n$.

Proof. We may assume $f \neq 0$ and that $n < \infty$.

⇐: Suppose $f(b, t)$ is K-separable with rank n, i.e., $\text{rank}_K(f) = n < \infty$. Then applying Lemma 1(d), equation II, we may write

$$f(b, t) = \sum_{i=1}^{n} g_i(b) \cdot h_i(t), \tag{2}$$

where, say, $g_i(b) = f(b, t_i)$ and $[h_i(t), \cdots, h_n(t)] = [f(b_i, t) \cdots f(b_n, t)] \cdot (M^{-1})'$, i.e., range $g_i(b) \subseteq F$ and range $h_i(t) \subseteq F$. By Lemma 2, the g_i's and the h_i's are linearly independent, respectively, over K and hence over $F \subseteq K$. Thus by Lemma 1(b), rank $_F(f) = n < \infty$.

⇒: Suppose $\text{rank}_F(f) = n < \infty$. Then we may express f in the form of equation (I) where the conditions of Lemma 1(b) hold. In order to show that $\text{rank}_K(f) = n$, we need only show that these g_i's and h_i's, respectively, are linearly independent functions over K. We do this for the g_i's. Suppose $k_1, k_2, \cdots, k_n$ contained in K are such that

$$\sum_{i=1}^{n} k_i g_i(b) \equiv 0, \qquad b \text{ in } B. \tag{3}$$

By Lemma (B), there exist $b_1, b_2, \cdots, b_n$ in B so that $[g_i(b_j)]$, $i, j = 1, \cdots, n$, is a nonsingular matrix. Substituting these b_j's in equation (3), we deduce, by

Cramer's Rule for the solution of a system of linear equations, that $k_i=0$ for each i.

Before presenting examples of separable and inseparable functions, we quote an invaluable result.

LEMMA 3 (Vandermonde determinant, cf. [1] and [8]). *Let* $x_1, x_2, \cdots, x_n$ *be* n *values from a field* F, *and let*

$$D=\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \qquad \text{(Vandermonde matrix).}$$

Then (a) det $(D)=\Pi(x_k-x_j)$, $1\leqq j<k\leqq n$;

(b) *the* x_i's *are distinct*$\Leftrightarrow$*the columns of* D *are linearly independent.*

Example 1. $f(b, t)=b_0+b_1t+\cdots+b_{n-1}t^{n-1}$ where $b=(b_0, \cdots, b_{n-1})$, b_i's and t are real, is a separable function of rank n.

Proof. Let $g_i(b)=b_{i-1}(i=1, \cdots, n)$, $h_i(t)=t^{i-1}(i=1, \cdots, n)$. Then the g's are linearly independent, i.e.,

$$\sum_1^n c_ig_i(b) = 0 \qquad \text{all } b'\text{s} \Rightarrow c_i = 0 \quad (i = 1, \cdots, n).$$

For proof, take b with all components except the ith equal to zero. The linear independence of the h's follows by noting that they are linearly independent if their domain is restricted to n distinct t_i's, in particular, that the matrix $[h_j(t_i)]_{i=1}^{n}\,_{j=1}^{n}$ is the Vandermonde matrix, cf. Lemma 3(b). Thus by Lemma 1(b), since $f(b, t)=\Sigma g_i(b)h_i(t)$, we are done.

Example 2. $f(a, t)=1+at+a^2t^2+\cdots+a^{n-1}t^{n-1}$, where a and t are real, is separable of rank n. The proof is obvious from the previous example.

Example 3. $f(n, t)=t^n$ $(t^0\equiv 1)$, $n=0, 1, 2, \cdots$, and t takes countably many distinct values $t_1, t_2, \cdots, t_i, \cdots$ in a field F, is inseparable (i.e., rank$(t^n)=\infty$).

Proof. We note by Lemma 3(b) that the matrix $[(t_i)^j]$, $i=1, \cdots, m$, $j=0, \cdots, m-1$ has rank m for each positive integer m. But if f were separable, these ranks would have to be bounded by a fixed integer. (Cf. Lemma 1(c).)

For subsequent use, we now investigate rank properties of combinations of separable functions.

LEMMA 4. *Let* $f(b, t)$, $g(b, t)$ *be separable functions of ranks* m *and* n *respectively* $(m\geqq n)$, *defined for* b *in* B, t *in* T *and taking their values in the same field* F. *Then*

(a) *the functions* $f(b, t)\cdot g(b, t)$ *and* $f(b, t)+g(b, t)$ *are separable,*

(b) $0\leqq rank(f\cdot g)\leqq m\cdot n$,

(c) *for any never-vanishing function with rank one, say* $r(b)\cdot s(t)$, *we have* $rank(r(b)s(t)f(b, t))=rank\, f(b, t)$,

(d) $m-n\leqq rank(f+g)\leqq m+n$.

Proof. (a) and (b) follow easily from Definition 1 by observing the forms involved.

(c) Consider $f(b, t)$ as a matrix. $s(t)f(b, t)$ is the new matrix obtained by multiplying each column of $f(b, t)$ by a nonzero constant. Thus, since the spaces spanned by the columns of these two matrices have the same dimension, it follows by Lemma 1(e) that rank $(s(t)f(b,t)) = \text{rank} f(b,t)$. Similarly, rank $(r(b)s(t)f(b,t)) = \text{rank}(s(t)f(b,t))$. Thus we obtain (c).

(d) We note by (c) that $\text{rank}(g) = \text{rank}(-g)$. That $\text{rank}(f+g)$ is $\leqq m+n$ follows immediately from the form of $f+g$. Using these two facts, we obtain

$$m = \text{rank}(f) = \text{rank}[(f+g) - g] \leqq \text{rank}(f+g) + \text{rank}(-g) = \text{rank}(f+g) + n,$$

i.e., $m \leqq \text{rank}(f+g)+n$, from which the first inequality of (d) follows.

Although we shall not subsequently need it, we conclude this section with a theorem on the rank of the usual type matrix product.

THEOREM 2. *Let* (i) *$f(a, b)$, $g(b, t)$ be separable functions of ranks m and n, respectively, taking their values in the same field F, defined for a in A, b in B, and t in T,*

(ii) *$b_1, b_2, \cdots, b_k$ be a finite sequence of distinct b's,*

(iii) *$h(a, t)$ be defined equal to $\Sigma_{i-1}^{k} f(a, b_i)g(b_i,t)$. Then $\text{rank}(h) \leqq \min(k, m, n)$.*

Proof. Follows easily from Definition 2,

5. A general approach. The linear space spanned by a set of vectors $\{f_\alpha\}$ for $\alpha \in A$ over a field F shall be denoted by $\mathcal{G}_F\{f_\alpha\}_{\alpha\in A}$. The following theorem gives us a general method of attack for proving a function is inseparable.

THEOREM 3. *Let* (i) *$f(b, t)$ be a function from $B \times T$ into a field F,*

(ii) *$\mathcal{L}$ be an additive homogeneous (i.e., linear) mapping from $\mathcal{G}_F\{f(\cdot, t)\}_{t\in T}$ into a vector space of F-valued functions $g(a)$ defined on a set A,*

(iii) *$\mathcal{L}(f(\cdot, t)) = g(\cdot, t)$.*

Then

(4) *$g(a, t)$ is inseparable $\Rightarrow f(b, t)$ is inseparable.*

Proof. The contrapositive form of (4) is

(4′) $f(b, t)$ is separable $\Rightarrow g(a, t)$ is separable.

From Lemma 1(e), it follows that (4′) is equivalent to

$$\text{(5)} \qquad \text{dimension } \mathcal{G}\{f\cdot, t)\}_{t\in T} < \infty \Rightarrow \text{dimension } \mathcal{G}\{g(\cdot, t)\}_{t\in T} < \infty.$$

We prove (5). Let $W = \mathcal{G}\{f(\cdot, t)\}_{t\in T}$. Then $\mathcal{L}(W) = \mathcal{G}\{g(\cdot, t)\}_{t\in T}$. Suppose $\dim(W) < \infty$, in particular that W is spanned by $f_1(b), \cdots, f_n(b)$. Since $\mathcal{L}(\Sigma_1^n c_i f_i(b) = \Sigma_1^n c_i \mathcal{L}(f_i(b)))$, it follows that any element in $\mathcal{L}(W)$ is a linear combination of the $\mathcal{L}(f_i)$ $(i = 1, \cdots, n)$, i.e., $\dim(\mathcal{L}(W))$ is finite.

Clearly, in applications, we shall always use an infinite set for A. For, if A were finite, $g(a, t)$ would always be separable (cf. Lemma 1(f)). Most often it will be reasonable to take $A = \{0, 1, 2, \cdots\}$, where $g(a)$ will be a sequence $(g_0, g_1, g_2, \cdots)$. $\mathcal{L}$ generally will be some sort of differentiation, indefinite integration, or (infinite)-matrix multiplication (cf. Theorem 2). We illustrate the use of Theorem 3 with two examples.

Example 4. $f(b, t)=e^{bt}$, b and t real, is inseparable.

Proof. Let $\mathcal{L}(f(\cdot, t))=D(\cdot, t)=D(t)$, where $D(t)$ is the sequence of derivatives (starting with the zero-th) of $f(\cdot, t)$ evaluated at $b=0$. Then $D(t)=(1, t, t^2, \cdots, t^n, \cdots)$, i.e., $D(n, t)=t^n$, $n=0, 1, 2, \cdots$. By Example 3, t^n is inseparable and thus, by Theorem 3, e^{bt} is inseparable.

Example 5. $f(b, t)=\sqrt{(b+t)}$, $b>0$, $t>0$ is inseparable.

Proof. Let $b_0>0$ be a fixed number. Similarly to Example 4, let $\mathcal{L}(f(\cdot, t))=D(\cdot, t)=D(t)$ where $D(t)$ is the sequence of derivatives of $f(\cdot, t)$ evaluated at $b=b_0$. Then

$$D(t) = ((b_0+t)^{1/2}, \tfrac{1}{2}(b_0+t)^{-1/2}, -\tfrac{1}{4}(b_0+t)^{-3/2}, \cdots, c(n)(b_0+t)^{1/2}(1/(b_0+t)^n_1, \cdots)$$

i.e., $D(n, t) = c(n)(b_0+t)^{1/2}(1/(b_0+t))^n = c(n)\cdot d(t)\cdot F(n, t)$, where $F(n, t)=(1/(b_0+t))^n$. By Lemma 4(c), D is inseparable if F is inseparable. But the inseparability of F is easily established by noting that $s=1/(b_0+t)$ runs through infinitely many distinct values and that s^n is inseparable (cf. Example 3).

6. Polynomial quotients. Here (Theorem 4) we shall completely solve the problem of the separability of quotients of polynomial functions. Our approach is motivated by the following.

Example 6. If $f(b,t)=1/(b-t)$, where b and t each take countably many values in an arbitrary field, then f is inseparable.

Proof (by contradiction). Suppose f is separable, say of rank m. We note that $g_n(b, t)=b^n-t^n$ is separable of rank 2 for each $n\geqq 1$. By Lemma 4(b), we have, for each $n\geqq 1$, $\text{rank}(f\cdot g_n)\leqq \text{rank}(f)\cdot\text{rank}(g_n)=2m$. But $f(b, t)\cdot g_n(b, t)=(b^n-t^n)/(b-t)=b^{n-1}\cdot 1+b^{n-2}\cdot t+\cdots+bt^{n-2}+1\cdot t^{n-1}$ has rank n since $\{b^i\}$, $i=0, \cdots, n-1$ and $\{t^i\}$, $i=0, \cdots, n-1$, are each sets of linearly independent functions (cf. Example 1 and Lemma 1(b)). Choosing $n=2m+1$, we have the contradiction $2m+1\leqq 2m$.

For subsequent use we generalize Example 4 into the following obvious

LEMMA 5. *Let $f(b, t)=1/(b-\beta(t))$ be defined for countably many b's in F and countably many values of t so that $\beta(t)$ takes countably many values in F. Then $1/(b-\beta(t))$ is inseparable.*

The following lemma emphasizes the fact that the rank of a function in polynomial form basically depends on the polynomial form.

LEMMA 6. *Let $f(b, t)$ be in polynomial form in t, b over a field F, defined for b and t in countably infinite subsets B and T, respectively, of F. Then the rank of $f(b, t)$ is independent of the particular subsets B and T which are used.*

Proof. Trivial, if $f\equiv 0$. Otherwise, suppose $\text{rank}(f)=n$. By Lemma 1, equation (II), we may write $f(b, t) = \Sigma_{i=1}^{n} g_i(b)h_i(t)$ where the g's and h's, respectively, are linearly independent functions in polynomial form. The "g's are linearly independent for b in B" means

$$(6) \qquad \sum_{i=1}^{n} c_i g_i(b) \equiv 0 \qquad \text{all } b \text{ in } B \Rightarrow c_i = 0 \quad (i = 1, \cdots, n).$$

Since a polynomial which vanishes for infinitely many variable (b)-values must be the zero polynomial, (6) says that the polynomials in b, $g_i(b)$ $(i=1, \cdots, n)$ are linearly independent. Clearly, the g's would be linearly independent, considered as functions over any infinite domain B. Similar reasoning holds with respect to the h's. Thus, by Lemma 1(b), we are done. We shall need the following obvious result for Theorem 4.

LEMMA 7. *Let $f(b, t)$ be defined for b in B and t in T. Let B' be contained in B and T' be contained in T. Then*

$$\text{rank } (f) = \infty,\ (b, t) \text{ in } B' \times T' \Rightarrow \text{rank } (f) = \infty,\ (b, t) \text{ in } B \times T.$$

We are now almost ready to prove Theorem 4, but first we need a lemma on factoring.

LEMMA 8. *Let* (i)$f(b, t) = \Sigma_{i=0}^{N} h_i(t)b^i = \Sigma_{i=0}^{M} r_i(b)t^i$, *where* $f \not\equiv 0$ *and the* h_i*'s and* r_i*'s are in polynomial form over a field* F, *be defined for countably many* b*'s and* t*'s respectively, in* F,

(ii) $h_N(t)$ *and* $r_M(b)$ *be nonvanishing over their domains,*

(iii) *the field* F *be extended* (cf. [1] p. 428) *to a larger field* K, *if necessary, so that in* K *for each* t, f *splits into a product of linear factors* $f(b, t) = f_t(b) = h_N(t)[(b-\alpha_1(t))(b-\alpha_2(t)) \cdots (b-\alpha_N(t))]$,

(iv) $l(b)$ *be the greatest common polynomial divisor of the coefficients* $r_i(b)$ (cf. [1] p. 396, Theorem 2).

Then

$$(7) \qquad \begin{cases} f(b, t) = l(b)\cdot h(b, t) \\ \qquad\quad = l(b)\cdot h_N(t)(b - \beta_1(t))(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t)), \end{cases}$$

where: (A) *rank*$(f) =$ *rank*(h); (B) $r =$ *degree of* $l(b)$; *and* (C) *none of the* $\beta_i(t)$ *take the same value more than* M *times.*

Proof. (A) follows immediately from Lemma 4(c). (B) is clear. It remains to prove (C): We note by (i) and (iv) that $h(b, t)$ may be written as

$$(8) \qquad h(b, t) = \sum_{i=0}^{M} S_i(b)t^i,$$

where the greatest common divisor of the polynomials $S_i(b)$ is a constant. If one of the $\beta_i(t)$'s takes the same value β_0 for more than M distinct t's, then it follows that the polynomial in t, $h(\beta_0, t)$ must be identically zero, i.e., $S_i(\beta_0) = 0$ for each i. But this implies that the $S_i(b)$'s have a nontrivial polynomial factor, contradicting (iv).

Now we prove the main result of this section.

THEOREM 4 (Polynomial Quotients). *Let* (i) $f(b, t)$ *and* $g(b, t)$, *in polynomial form in* b, t *over a field* F, *be defined for infinitely many* b'*s and* t'*s respectively in* F,

(ii) $rank(f) \geqq 2$, $rank(g) \geqq 1$, $f \equiv nonzero$,

(iii) $f(b, t) = l(b) \cdot h(b, t)$ *be the factorization of* f *in accordance with Lemma* 8, *Equation* (7).

Then $rank(g(b, t)/f(b, t)) = \infty \Leftrightarrow$ *the polynomial* $h_t(b) = h(b, t)$ *does not divide* $g_t(b) = g(b, t)$ *exactly* (*both considered as polynomials in the one variable* b).

Proof. $\Rightarrow$: The contrapositive of this implication is "$h_t(b)$ divides $g_t(b)$ exactly $\Rightarrow g(b, t)/f(b, t)$ is separable." We show this. By hypothesis

$$g_t(b)/h_t(b) = \sum_{i=0}^{m} p_i(t) b^i \tag{9}$$

where the $p_i(t)$ are rational fractional forms in t over F. Thus it follows that

$$g_t(b)/f_t(b) = \sum_{i=1}^{m} \{p_i(t)(b^i/l(b))\} \text{ is separable.} \tag{10}$$

$\Leftarrow$: In this case

$$g_t(b)/h_t(b) = \sum_{i=1}^{n} p_i(t) b^i + \gamma_t(b)/h_t(b), \tag{11}$$

where $\gamma_t(b)$ is of lower degree in b than $h_t(b)$ and has rational fractional forms in t as coefficients. Without loss of generality (cf. Lemma 7) we may assume that Equation (7) of Lemma 8 holds, i.e., $h_t(b) = h_N(t)(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t)) = h_N(t) q(b, t)$.

We shall show that $\gamma_t(b)/q(b, t)$ is inseparable. From this by use of Lemma 4(c) and the easily proven fact that the sum of a separable function and an inseparable function is inseparable, it will immediately follow that $g(b, t)/f(b, t)$ is inseparable.

First, divide the polynomial in b, $b - \beta_1(t)$, into $\gamma_t(b)$, getting

$$\frac{\gamma_t(b)}{b - \beta_1(t)} = Q(b, t) + \frac{r_1(t)}{b - \beta_1(t)}; \tag{12}$$

hence

$$\frac{\gamma_t(b)}{q(b, t)} = \frac{r_1(t)}{(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t)} + \frac{Q(b, t)}{(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t))}. \tag{13}$$

Next, divide $Q(b, t) = Q_t(b)$ by $b - \beta_2(t)$, etc., finally obtaining

$$\begin{aligned} \frac{\gamma_t(b)}{q(b, t)} = \frac{r_1(t)}{(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t))} + \frac{r_2(t)}{(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t))} + \cdots \\ + \frac{r_{N-r}(t)}{b - \beta_{N-r}(t)}. \end{aligned} \tag{14}$$

We claim that at least one of the functions $r_i(t)$ must be nonzero for infinitely many of our t's. For otherwise $\gamma_t(b)$ would be zero for a fixed b and infinitely many t's and thus would be identically zero, contradicting our hypothesis. Now let j be the smallest integer so that $r_j(t)$ is nonzero infinitely often. Without loss of generality (cf. Lemma 7) we may discard the finitely many t-values for which $r_i(t)$, $i<j$, do not vanish. So

$$\text{(15)} \qquad \frac{\gamma_t(b)}{q(b,t)} = \frac{r_j(t)}{(b-\beta_j(t))\cdots(b-\beta_{N-r}(t))} + \cdots + \frac{r_{N-r}(t)}{b-\beta_{N-r}(t)}\cdot$$

We show (15) is inseparable by contradiction. Note that

$$\text{(16)} \qquad k(b,t) = (b-\beta_{j+1}(t))\cdot(b-\beta_{j+2}(t))\cdots(b-\beta_{N-r}(t))$$

is separable. Hence if (15) were separable, then the product of (15) and (16) would be separable. But

$$\text{(17)} \qquad \begin{aligned} k(b,t)\cdot\frac{\gamma_t(b)}{q(b_1t)} = \frac{r_j(t)}{b-\beta_j(t)} &+ r_{j+1}(t) + \cdots \\ &+ r_{N-r}(t)(b-\beta_{j+1}(t))\cdots(b-\beta_{N-r+1}(t)), \end{aligned}$$

where by Lemma 5, the first term on the right hand side of (17) is inseparable and hence (17) is inseparable.

We point out, in our proof of Theorem 4, our assumption that $\operatorname{rank}(f)\geqq 2$ forces the existence of at least one $\beta_i(t)$ (cf. (12) and (13)).

Although our hypotheses in Theorem 2 are in a one-sided form, we state the nicely symmetric

COROLLARY. *Let $f(b, t)$ be in polynomial form over the field F, defined for infinitely many b's and t's respectively, in F, and not vanish. Then $\operatorname{rank}(f)\geqq 2$ $\Leftrightarrow 1/f(b, t)$ is inseparable.*

We note immediately from the corollary that forms such as

$$1/(b\cdot t^2 + 1\cdot t + b^2\cdot 1), \qquad 1/(b^2+t^3).$$

etc., will give us inseparable functions if we can choose infinite sequences of distinct b's and t's, respectively, from a field F so that the denominators will not vanish for any pair (b, t). If F is an infinite field, this may always be done inductively: First choose a b_1 so that $f(b_1, t)\not\equiv 0$. Then choose t_1 so that $f(b_1,t_1)\neq 0$. Then choose b_2 so that $f(b_2, t_1)\neq 0$. Then choose t_2, etc. These choices always exist since a polynomial in one variable can have at most a finite number of distinct roots.

7. Pureness. Let rank $(f)=n$, $1\leqq n\leqq\infty$. We say that f has *pure rank* n if every square $m\times m$ matrix $[f(b_i, t_j)]$, $i, j=1, \cdots, m$ where the b_i's and t_i's respectively run through distinct values, and where m is finite, $\leqq n$, has rank m (i.e., is of full rank).

Example 7. $f(b, t)=1/(b+t)$, where b and t respectively take countably many values from a field F, has pure rank $n=\infty$.

Proof. We refer to the following theorem due to A. Cauchy (cf. [4] [11]).

THEOREM. *Let $b_1, \cdots, b_m, t_1, \cdots, t_m$ be 2m numbers. Then*

$$\det\left[\frac{1}{b_i + t_j}\right]_{i=1}^{m}{}_{j=1}^{m} = \frac{\prod_{1 \leq i < j \leq m} (b_j - b_i)(t_j - t_i)}{\prod_{i,j=1}^{m} (b_i + t_j)}.$$

Clearly, for distinct b's and t's, it follows that $\det [1/(b_i+t_j)] \neq 0$ and thus that the matrix $[1/(b_i+t_j)]$ has rank m for each $m < \infty$.

With the posing of the general question, "Which functions have pure rank?" we end this section.

8. Conclusion. Although we have assumed a previous knowledge of matrices in the development of this paper, it is obvious that this is not necessary and that one may make our Definitions 1 and 2 the basis for the development of properties of matrices. Indeed the symmetry of our approach makes it seem more appealing than the usual definition of matricial rank in terms of row or column rank. We close with the hope that perhaps our small store of results also clears up various problems in the possibilities of approximating one function of two variables by another.

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ON POINT TRANSFORMATIONS

JOSEPH VERDINA, California State College at Long Beach

Transformations which send points into points were studied by Möbius, Chasles, Poncelet and others. In this paper some definitions and properties of such transformations are recalled. Subsequently some notions of a local study of point transformations in the vicinity of a regular pair of corresponding points are introduced. Such studies have been the object of recent investigations. Finally, a theorem is stated which brings to light an interesting property of the quadratic transformations. Our discussion is limited to plane transformations over the complex field.

1. A point transformation may be defined by the equations

$$\rho y_i = f_i(x_1, x_2, x_3) \qquad i = 1, 2, 3$$

where the x_i and y_i are homogeneous coordinates in the field of complex numbers and the f_i are homogeneous continuously differentiable functions for which the Jacobian $|\partial f_i/\partial x_j|$ $i, j = 1, 2, 3$ does not identically vanish. The collineations are point transformations. If θ and θ' are two projective planes, the equations

$$\rho y_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \qquad i = 1, 2, 3 \tag{1}$$

where the a are constants for which $|a_{ij}| \neq 0$ and ρ is a constant different from zero, determine a correspondence called *collineation* between the two planes. Such planes may be distinct or superposed. Equations (1) are called *equations* of the collineation. By letting $x = (x_1/x_3)$, $y = (x_2/x_3)$, $x' = (y_1/y_3)$, $y' = (y_2/y_3)$ we get equations in nonhomogeneous coordinates. Division of the first two of (1) by the third gives

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \qquad y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}. \tag{2}$$

In a collineation a line maps into a line.

2. Correlations. A transformation defined by

$$p_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3, \qquad i = 1, 2, 3$$

where the x_i are point coordinates and the p_i are line coordinates, is called a *correlation*. It transforms a point into a line. If the determinant $|a_{ij}|$ is equal to zero, then the correlation is called *degenerate*. The equations of a correlation may be written in the form

$$(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)y_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)y_2 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)y_3 = 0.$$

3. Quadratic transformations. We have seen that the collineations establish a one-to-one algebraic relation of the first degree between the coordinates of two corresponding points. The following question may be posed: Are there transformations which preserve the essential characters of the collineation—that is, of being algebraic and one-to-one between points—but without having the form of polynomials of the first degree? The answer is affirmative. Such transformations exist and are called birational or Cremona transformations from the name of the geometer, Cremona (1830–1903), who was the first to study them in their full generality. The equations of a Cremona transformation are of the kind

$$\begin{aligned} \rho x_1 &= \psi_1(y_1, y_2, y_3) \\ \rho x_2 &= \psi_2(y_1, y_2, y_3) \\ \rho x_3 &= \psi_3(y_1, y_2, y_3) \end{aligned} \tag{3}$$

where the ψ_i are polynomials. If the ψ_i are quadratic polynomials, equations (3) are the equations of a quadratic transformation. Solved for y_i they give

$$\begin{aligned} \sigma y_1 &= \phi_1(x_1, x_2, x_3) \\ \sigma y_2 &= \phi_2(x_1, x_2, x_3) \\ \sigma y_3 &= \phi_3(x_1, x_2, x_3). \end{aligned} \tag{4}$$

4. A quadratic transformation between two planes θ and θ' may also be defined as the intersection of two correlations, say R_1 and R_2. Its equations have the form

$$\begin{aligned} &(a_{11}x_1 + a_{12}x_2 + a_{13}x_3)y_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)y_2 \\ &\qquad\qquad + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)\, y_3 = 0. \\ &(b_{11}x_1 + b_{12}x_2 + b_{13}x_3)y_1 + (b_{21}x_1 + b_{22}x_2 + b_{23}x_3)y_2 \\ &\qquad\qquad + (b_{31}x_1 + b_{32}x_2 + b_{33}x_3)\, y_3 = 0. \end{aligned} \tag{5}$$

By letting $(x_1/x_3) = x$, $(x_2/x_3) = y$, $(y_1/y_2) = x'$, $(y_2/y_3) = y'$, equations (5) may be written in nonhomogeneous form as

$$\begin{aligned} (a_{11}x + a_{12}y + a_{13})x' + (a_{21}x + a_{22}y + a_{23})y' + (a_{31}x + a_{32}y + a_{33}) &= 0. \\ (b_{11}x + b_{12}y + b_{13})x' + (b_{21}x + b_{22}y + b_{23})y' + (b_{31}x + b_{32}y + b_{33}) &= 0, \end{aligned} \tag{6}$$

or, in more concise form, as

$$\begin{aligned} u_1x' + u_2y' + u_3 &= 0 \\ v_1x' + v_2y' + v_3 &= 0, \end{aligned} \tag{7}$$

where the u_i and v_i denote linear polynomials in x and y. Solving for x' and y' we obtain

$$x' = \frac{f_1(x, y)}{f_3(x, y)} \qquad y' = \frac{f_2(x, y)}{f_3(x, y)}, \tag{8}$$

where the f_i are quadratic polynomials. Equations (8) written in homogeneous form take the form of equations (4).

Consider now a point P in the plane θ and let the transform of P under the correlation R_1 be the line r_1 and under R_2 the line r_2. The lines r_1 and r_2 intersect at a point P' unless r_1 and r_2 are coincident. Thus, with an arbitrary point P in θ is associated a point P' in θ', and conversely, with some possible exceptions. These exceptional points are called *singular*.

We now investigate the number of singular points of the transformation. Let the equations of a quadratic transformation be

$$
\begin{aligned}
\rho x_1 &= \psi_1(y_1, y_2, y_3)\\
\rho x_2 &= \psi_2(y_1, y_2, y_3)\\
\rho x_3 &= \psi_3(y_1, y_2, y_3)
\end{aligned}
\tag{9}
$$

where ρ is a proportionality factor and the ψ_i are quadratic polynomials. Solved for y, equations (9) must give equations (4).

Since the ψ_i are of the second order, how must they be chosen to give equations (4)?

Solving equation (9) is equivalent to determining in θ' the points common to the two conics

$$
\begin{aligned}
x_3\psi_1(y_1, y_2, y_3) - x_1\psi_3(y_1, y_2, y_3) &= 0\\
x_3\psi_2(y_1, y_2, y_3) - x_2\psi_3(y_1, y_2, y_3) &= 0
\end{aligned}
\tag{10}
$$

obtained by elimination of the factor ρ. The number of these points is in general four and if all four had the coordinates dependent on those of $P \equiv (x_1, x_2, x_3)$ the correspondence defined by (9) would not be one-to-one. Hence three of these points, say A_1, A_2, A_3, must be fixed, and then the fourth, $P' \equiv (y_1, y_2, y_3)$, is the transform of $P \equiv (x_1, x_2, x_3)$. The points, A_1, A_2, A_3, are the singular points of the transformation. Their number is three and they must satisfy the three equations

$$\psi_1 = 0, \qquad \psi_2 = 0, \qquad \psi_3 = 0.$$

Example. The singular points of the inversion are the center of inversion and the cyclic points of the plane. In fact, the Cartesian equations of the inversion

$$x' = \frac{kx}{x^2 + y^2}, \qquad y' = \frac{ky}{x^2 + y^2}$$

written in homogeneous form are

$$y_1 = kx_1x_3, \qquad y_2 = kx_2x_3, \qquad y_3 = x_1^2 + x_2^2.$$

Solving the system

$$
\begin{aligned}
kx_1x_3 &= 0\\
kx_2x_3 &= 0\\
x_1^2 + x_2^2 &= 0
\end{aligned}
$$

we get the points $(0, 0, 1)$, $(1, i, 0)$ and $(1, -i, 0)$.

PROPOSITION 1. *Under a quadratic transformation between two planes θ and θ', if a point P of θ describes a line, its transform P' in θ' describes a conic.*

In fact, if the point P in θ describes the line

$$\lambda x_1 + \mu x_2 + \nu x_3 = 0,$$

by equations (9), its transform P' in θ' describes the curve

$$\lambda\psi_1(y_1, y_2, y_3) + \mu\psi_2(y_1, y_2, y_3) + \nu\psi_3(y_1, y_2, y_3) = 0. \tag{11}$$

Not only the conics $\psi_1=0$, $\psi_2=0$, $\psi_3=0$ go through A_1, A_2, A_3 but also all the other conics of the net represented by (11) as the ratios $\lambda:\mu:\nu$ vary, that is, as the line (3) in θ varies. This net of conics is called *homoloidal* and it is such that any two conics of the net have only one variable intersection.

Let A_1, A_2, A_3 be the singular points of the quadratic transformation T_2 in the plane θ and let a_1', a_2', a_3' be the singular lines in θ' associated with the points A_1, A_2, A_3 under T_2. The point A_3', intersection of a_1' and a_2', is a singular point since it is associated with A_1 and A_2. Two points such as A_3 and A_3' are said to be *associate*.

We recall the following property:

PROPOSITION 2. *Under a quadratic transformation, between two planes θ and θ', a line r of θ through a singular point A_3 is transformed into a degenerate conic formed by the singular line a_3', associated with A_3, and by a line r' through the singular point A_3' associated with A_3.*

Example. Let the equations of a quadratic transformation between two planes θ and θ' be

$$\begin{aligned} x_1 &= \tau y_2 y_3 \\ x_2 &= \tau y_1 y_3 \\ x_3 &= \tau y_1 y_2. \end{aligned} \tag{12}$$

We verify that the singular points of the transformation are $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$. The line $x_1+kx_2=0$ through $A_3(0, 0, 1)$, by equations (12), is transformed into the degenerate conic $y_3(y_2+ky_1)=0$ which is composed of the line $y_3=0$ and the line $y_2+ky_1=0$ through $A_3'(0, 0, 1)$.

5. Local point transformation. Let x_1, x_2 and y_1, y_2 be nonhomogeneous Cartesian or projective coordinates in two planes θ and θ'. A point transformation T between θ and θ' may be expressed by

$$y_i = f_i(x_1, x_2) \qquad i = 1, 2$$

where f_1 and f_2 are functions of the two real variables x_1 and x_2, defined in the given region of θ, and expressible in power series in the neighborhood of the point O (of coordinates $\bar{x}_1$, $\bar{x}_2$). Let $O'(a_0, b_0)$ be the corresponding point of O under the transformation T.

The series

$$\begin{aligned} y_1 = f_1(x_1, x_2) &= a_0 + a_1(x_1 - \bar{x}_1) + a_2(x_2 - \bar{x}_2) + a_{11}(x_1 - \bar{x}_1)^2 \\ &\quad + 2a_{12}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + a_{22}(x_2 - \bar{x}_2)^2 + \cdots \\ y_2 = f_2(x_1, x_2) &= b_0 + b_1(x_1 - \bar{x}_1) + b_2(x_2 - \bar{x}_2) + b_{11}(x_1 - \bar{x}_1)^2 \\ &\quad + 2b_{12}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) + b_{22}(x_2 - \bar{x}_2)^2 + \cdots \end{aligned} \tag{13}$$

where

$$a_1 = \frac{\partial f_1}{\partial x_1}, \qquad a_2 = \frac{\partial f_1}{\partial x_2}, \qquad a_{11} = \frac{1}{2}\frac{\partial^2 f_1}{\partial x_1^2}, \cdots, b_1 = \frac{\partial f_2}{\partial x_1}$$

and the derivatives are evaluated at $x = \bar{x}_1$, $x_2 = \bar{x}_2$, converge when real or complex values are assigned to the variables x_1 and x_2, provided that they are close to $\bar{x}_1$ and $\bar{x}_2$. The rth-order neighborhood of (O, O') is determined by the equations obtained from (13) by deleting, in the series, the terms following those of order r in $x-\bar{x}$, $y-\bar{y}$. The local study of a point transformation considers the transformation in the successive neighborhoods of the pair (O, O'). When the Jacobian of f_1, f_2, calculated at $\bar{x}$, $\bar{y}$, is different from zero, the pair (O, O') is said to be regular.

6. Projectivities between directional pencils with centers at two corresponding points. Let O, O' be the origins of the coordinates. Equations (13) of T, in the neighborhood of (O, O'), become

$$\begin{aligned} y_1 &= a_1x_1 + a_2x_2 + R_2 \\ y_2 &= b_1x_1 + b_2x_2 + R_2 \end{aligned} \tag{14}$$

where we have denoted by R_2 all terms in x_1, x_2 of degree $\geqq 2$. To a line through O'

$$y_1 + \lambda y_2 = 0 \tag{15}$$

there corresponds, under the point transformation T of equations (14), the curve of equation

$$a_1x_1 + a_2x_2 + \lambda(b_1x_1 + b_2x_2) + R_2 = 0,$$

whose tangent at O is the line

$$a_1x_1 + a_2x_2 + \lambda(b_1x_1 + b_2x_2) = 0. \tag{16}$$

Let us suppose now that the pair (O, O') be regular. If we make a line through the point O' given by (15) correspond to the tangent to the corresponding curve at O given by (16), we define a projectivity, say ω, inasmuch as the pencils (15) and (16) are projective and the lines given by the same value of the coordinate λ are corresponding lines.

Let us assume now as lines $y_1 = 0$, $y_2 = 0$, $y_1 = y_2$ the lines that correspond, under ω, to $x_1 = 0$, $x_2 = 0$, $x_1 = x_2$, respectively. Then equations (13) yield $a_2 = b_1 = 0$, $a_1 = b_2$. By setting $a_1 = \alpha$, the T in the neighborhood of (O, O') is represented by

$$y_1 = \alpha x_1 + R_2, \qquad y_2 = \alpha x_2 + R_2,$$

where α is a constant $\neq 0$.

7. The characteristic directions. Passing to the second-order neighborhood of the regular pair (O, O'), under the transformation T, equations (13) of T may be written in the following form:

$$
(17) \qquad \begin{aligned} y_1 &= \alpha x_1 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + R_3 \\ y_2 &= \alpha x_2 + b_{11}x_1^2 + 2b_{12}x_1x_2 + b_{22}x_2^2 + R_3, \end{aligned}
$$

where by R_3 we denote the terms in x_1, x_2 of degree $\geqq 3$. The corresponding curve to the line through O' given by (15) is therefore

$$
(18) \quad \alpha(x_1 + \lambda x_2) + (a_{11} + \lambda b_{11})x_1^2 + 2(a_{12} + \lambda b_{12})x_1x_2 + (a_{22} + \lambda b_{22})x_2^2 + R_3 = 0.
$$

The following question may now arise: are there some lines given by (15) whose corresponding curves (18) have at O an inflection? And if there are such lines, which ones are they?

The point O is a point of inflection of the curve given by (18) if and only if the tangent at O, namely $x_1+\lambda x_2=0$, has at O a contact of the second order with the curve itself. Setting $x_1=-\lambda x_2$ in (18), and equating to zero the coefficient of x_2^2, one obtains

$$
(19) \qquad b_{11}\lambda^3 - (2b_{12} - a_{11})\lambda^2 + (b_{22} - 2a_{12})\lambda + a_{22} = 0.
$$

The lines (15) for which the curve (18) has at O an inflection are therefore those for which λ satisfies (19), that is, the lines

$$
(20) \qquad b_{11}y_1^3 + (2b_{12} - a_{11})y_1^2y_2 + (b_{22} - 2a_{12})y_1y_2^2 - a_{22}y_2^3 = 0.
$$

Excluding the case when (20) reduces to an identity—namely, the case when all the lines through O' satisfy the required condition—we conclude that the required lines are three and they are represented by (20). Evidently, these lines may be real or imaginary, distinct or coincident.

The inflectional tangents at O to the three curves, corresponding under T to the three lines (20), are the lines through O such that the corresponding curves under T have an inflection in O'. Such lines through O, being the corresponding lines of (20) under ω, have equation

$$
(21) \qquad b_{11}x_1^3 + (2b_{12} - a_{11})x_1^2x_2 + (b_{22} - 2a_{12})x_1x_2^2 - a_{22}x_2^3 = 0.
$$

The lines through O, such that the corresponding curve under T has in O' an inflection, are called *characteristic or inflectional lines*. The relative directions are called *characteristic directions*. We observe that under a quadratic transformation the pairs with Jacobian equal to zero are those formed by a point P on a singular line a (different from the singular points on a) and by the singular point associated with the line a. If a point O in θ is not on a side of the triangle of the singular points, then the corresponding point O' is also not on a side of the triangle of the singular points of θ' and the pair (O, O') is regular.

The following theorem ties the previous result with the Proposition 2 relative to the quadratic transformations.

THEOREM. *Under a quadratic transformation between two planes with (O, O') a regular pair of corresponding points, the lines joining O with the three singular points are the characteristic lines through O.*

Proof. Let A_1 be a singular point; under the quadratic transformation a line

OA_1 is transformed into the degenerate conic composed of two lines, one of which is the singular line a_1' which corresponds to A_1 and the other is a line through the origin O'. This line through O' is the line $O'A_1'$, where A_1' is the singular point associated with A_1. Hence the transform of a line OA_1 is a curve with an inflection at O'.

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THE CROSS RATIO ON THE REAL LINE

EARL GLEN WHITEHEAD, JR., California Institute of Technology

1. Introduction. The cross ratio is defined as follows:

$$CR(a, b, c, d) = \frac{d-a}{d-b}\,\frac{c-b}{c-a}\,.$$

Bergquist and Foster [1] consider representations of the form $a^2+b^2+c^2+d^2$ where $CR(a^2, b^2, c^2, d^2) = n$ for all $n \in Z$ (the integers). They call an integer which has such a representation with $n = -1$, *a harmonic integer*. They show that there are infinite classes of harmonic integers and that there is at least one integer which has such a representation for all $n \in Z$. Coxeter [2] shows how to construct a fourth point given three distinct points on a line such that the cross ratio of these points is minus one.

2. Minus one cross ratio. Dr. J. W. Bergquist showed the author the following:

$$61 = 6^2 + 4^2 + 3^2 + 0^2,\ CR(6, 3, 4, 0) = -1,\ 61 = 7^2 + 3(2^2), \quad \text{and}$$
$$6 + 4 + 3 + 0 = 7 + 3(2).$$

The author generalized this phenomenon into the following theorem where a, b, c, d, e, f, $l \in Z$:

THEOREM. *If $l = a^2+b^2+c^2+d^2$ where $a>b>c>d$ and $CR(a, c, b, d) = -1$, then there are $e, f \in Z$ such that $l = e^2+3f^2$ where $a+b+c+d = e+3f$ and l is a quadratic residue modulo six.*

Proof. The cross ratio of numbers on the real line does not depend upon the size of these numbers, but upon the ratios of their differences. The following notation will prove to be useful:

$$a = R + d, \quad b = S + d, \quad c = N + d, \quad e = X + d, \quad \text{and} \quad f = Y + d$$
$$\text{where} \quad R, S, N, X, Y \in Z.$$

Now $a + b + c + d = e + 3f \Leftrightarrow R + S + N = X + 3Y$ (A) and

$$a^2 + b^2 + c^2 + d^2 = e^2 + 3f^2$$

$\Leftrightarrow R^2+S^2+N^2+2(R+S+N)d=X^2+3Y^2+2(X+3Y)d \Leftrightarrow R^2+S^2+N^2=X^2+3Y^2$ (B) assuming equation (A). The square of equation (A), $(R+S+N)^2=(X+3Y)^2$, minus equation (B) yields: $2(RS+SN+RN)=6XY+6Y^2 \Rightarrow RS+SN+RN=3Y(X+Y)$ (C); equation $A \Rightarrow X+Y=R+S+N-2Y$ (D). Combining equations (C) and (D) yields: $RS+SN+RN=3(R+S+N)Y-6Y^2$ (E). Now

$$CR(a,c,b,d)=\frac{d-(R+d)}{d-(N+d)}\cdot\frac{(S+d)-(N+d)}{(S+d)-(R+d)}=\frac{-R}{-N}\cdot\frac{S-N}{S-R}=-1$$

$\Rightarrow RS-RN=RN-SN \Rightarrow 2RN=RS+SN=S(R+N)$. By equation (E), $3RN=3(R+S+N)Y-6Y^2 \Rightarrow 2Y^2-(R+S+N)Y+RN=0$; thus

$$Y=\frac{R+S+N\pm\sqrt{(R+S+N)^2-4(2)RN}}{2(2)}$$

$$\begin{aligned}(R+S+N)^2-4(2)RN &= R^2+S^2+N^2+2RS+2RN+2SN-8\frac{RS+SN}{2}\\ &= R^2+S^2+N^2-2RS+2RN-2SN=(R-S+N)^2.\end{aligned}$$

Hence $Y=(R+S+N\pm(R-S+N))/4=(S/2)$ or $(R+N)/2$.

Since $2RN=S(R+N) \Rightarrow$ either S is even or $(R+N)$ is even, there is at least one choice of sign such that X and Y will be integers. Since $R+S+N=X+3Y$ we have integral X and Y in *either*

Case I: $Y=(S/2) \Rightarrow X=R-(S/2)+N$ *or*

Case II. $Y=(R+N)/2 \Rightarrow X=-(R/2)+S-(N/2)$ *or both.*

Recall that X and Y were found such that $l=a^2+b^2+c^2+d^2=e^2+3f^2$ and $a+b+c+d=e+3f$. Now the quadratic residues modulo six are 0, 1, 3, and 4. Hence $e^2, f^2 \equiv 0, 1, 3, 4 \bmod 6$ and $3f^2 \equiv 0, 3 \bmod 6$. Therefore we have $l=e^2+3f^2 \equiv 0, 1, 3, 4 \bmod 6$, i.e., l is a quadratic residue modulo six.

3. Infinitely many of such integers. An IBM 7094 program was written by the author to find all integers l such that $l=a^2+b^2+c^2+d^2$, where $1 \leqq l \leqq 1000$ and $CR(a, c, b, d)=-1$. This program first found all R, S, and N such that $45 \geqq R > S > N \geqq 1$ where $2RN=S(R+N)$, i.e., $CR(R, N, S, 0)=-1$. These values of R, S, and N are listed in Table 1.

TABLE 1

R	S	N	R	S	N	R	S	N	R	S	N
6	3	2	6	4	3	12	6	4	12	8	6
15	5	3	15	12	10	18	9	6	18	12	9
20	8	5	20	15	12	24	12	8	24	16	12
28	7	4	28	24	21	30	10	6	30	15	10
30	20	15	30	24	20	35	20	14	35	21	15
36	18	12	36	24	18	40	16	10	40	30	24
42	12	7	42	21	14	42	28	21	42	35	30
45	9	5	45	15	9	45	36	30	45	40	36

For each of these values of R, S, and N, values of $l=(R+d)^2+(S+d)^2+(N+d)^2+d^2$ where $-32\leqq d\leqq 32$ were computed. The twelve values of l found between 1 and 100 are as follows:

19	21	25	31	39	49
61	75	76	84	91	100

To indicate how these integers l are distributed from 1 to 1000, Table 2 is offered.

TABLE 2

Range from to	1 100	101 200	201 300	301 400	401 500	501 600	601 700	701 800	801 900	901 1000
Number of l's in range	12	18	18	18	15	17	20	16	18	19

THEOREM. *There are an infinite number of integers l such that $l=a^2+b^2+c^2+d^2$ where $CR(a, c, b, d)=-1$.*

Proof. Let $R=6n$, $S=3n$, and $N=2n$; thus $a=6n+d$, $b=3n+d$, and $c=2n+d$.

$$CR(a,c,b,d)=\frac{d-(6n+d)}{d-(2n+d)}\cdot\frac{(3n+d)-(2n+d)}{(3n+d)-(6n+d)}=\frac{-6n}{-2n}\cdot\frac{n}{-3n}=-1$$

By letting d range over all integers or by letting n range over all positive integers, an infinite number of integers $l=(6n+d)^2+(3n+d)^2+(2n+d)^2+d^2$ are generated.

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ON THE DIFFERENTIAL EQUATION $f'=f\circ g$ WHERE $g\circ g=I$

R. G. KULLER, Northern Illinois University

1. Introduction. In 1965 W. R. Utz [1] posed the problem of obtaining general results on the equation $f'=af\circ g$, noting that the equations $f'(x)=f(1/x)$ and $f^{(n)}=f(1/x)$ had been studied ([2] and [3]). In 1966 a general existence theorem for $f'(x)=F[x, f\circ g(x)]$ was given in [4] by Anderson, who later produced a related result in collaboration with Bogdanowicz [5].

This note deals with the special cases

$$f'=f\circ g, \tag{1}$$

and

$$f'' = f \circ g \tag{2}$$

where g is a solution of the functional equation

$$g \circ g = I. \tag{3}$$

(I is the identity function). The functions $g(x) = -x$ and $g(x) = x^{-1}$ are examples of solutions of (3), but in general any function g, which maps an interval J in a one to one manner onto J and which has a graph symmetric about the line $y = x$, also satisfies (3).

2. The functional equation $g \circ g = I$. We will need certain information about the solutions of (3).

THEOREM 1. *Let g be a continuous function on an interval J satisfying* (3), *and assume $g \neq I$. Then g is strictly decreasing and there is a unique $a \in J$ such that $g(a) = a$.*

The proof is accomplished by using the intermediate value theorem of elementary calculus. To establish the existence of the fixed point a, this theorem is applied to the function $g - I$. We omit the details.

COROLLARY. *If g is differentiable on J and if $g \circ g = I$, then $g' < 0$ on J and $g'(a) = -1$.*

Notice that the continuity of g' is not required. The mean value theorem shows that $g' < 0$ somewhere, while

$$(g' \circ g)g' = 1 \tag{4}$$

shows that g' is nowhere zero. Since any derivative has the intermediate value property, g' must be negative everywhere on J. An application of (4) at $x = a$ yields $g'(a) = -1$.

3. The equation $f' = f \circ g$. Let g be a differentiable solution of (3) $g \circ g = I$, $g \neq I$, and let f be a solution of (1) $f' = f \circ g$. Then if a is the fixed point of g, we have $f'(a) = f[g(a)] = f(a)$. Also, $f'' = (f' \circ g)g' = [(f \circ g) \circ g]g' = fg'$ so that $f'' - g'f = 0$. Thus we have proved half of the following theorem.

THEOREM 2. *If g is a differentiable solution of $g \circ g = I$, and $g \neq I$, then the following are equivalent:*

(a) $$f' = f \circ g$$

(b) $$\begin{cases} f'' - g'f = 0 \\ f'(a) = f(a), \end{cases}$$

where a is the unique fixed point of g.

It remains to be shown that (b) implies (a). Assume that f satisfies (b) and define $h = f' - f \circ g$. Then we have

$$\begin{aligned} h' &= f'' - (f'\circ g)g' \\ &= (f - f'\circ g)g' \\ &= -\,(h\circ g)g' \end{aligned}$$

and

$$\begin{aligned} h'\circ g &= (-h)(g'\circ g) \\ &= -\,h(g')^{-1}. \end{aligned}$$

Hence $h=-(h'\circ g)g'$ or $h=-(h\circ g)'$.

If we integrate from $g(x)$ to a, we find $\int_{g(x)}^{a} h\ dt=-h(a)+h(x)$. But $h(a)=f'(a)-f[g(a)]=f'(a)-f(a)=0$ and so the continuous function h satisfies the integral equation

$$h(x) = \int_{g(x)}^{a} h(t)dt. \tag{5}$$

If we iterate this equation we obtain

$$h(x) = \int_{g(x)}^{a}\left[\int_{g(t)}^{a} h(s)ds\right]dt.$$

Now, by interchanging the order of integration, we have

$$h(x) = -\int_{a}^{x} h(s)\left[\int_{g(x)}^{g(s)} dt\right]ds$$

or

$$h(x) = \int_{a}^{x} h(s)[g(x) - g(s)]ds. \tag{6}$$

(This is valid for all x in J.) Let $M_1=\text{Max}\ \{|h(s)| : a\leqq s\leqq x\}$ and $M_2=\text{Max}\ \{|g(x)-g(s)| : a\leqq s\leqq x\}$. From (6) it follows that $|h(x)|\leqq M_1M_2|x-a|$ and, by iteration,

$$\begin{aligned} |\,h(x)\,| &= \left|\int_{a}^{x} h(s)[g(x) - g(s)]ds\right| \\ &\leqq \left|\int_{a}^{x} M_1M_2\,|\,s - a\,|\,M_2ds\right| \end{aligned}$$

or

$$|\,h(x)\,| \leqq M_1M_2^2\,\frac{|\,x - a\,|^2}{2}\,.$$

By continuing to iterate we see that

$$|\,h(x)\,| \leqq M_1\frac{[M_2\,|\,x - a\,|\,]^n}{n!} \tag{7}$$

for all positive integers n. It follows that h is the zero function, or, in other words, f satisfies (a).

4. The equation $f''=f\circ g$. Again we are assuming that g is a differentiable solution of (3) $g\circ g=I$, $g\neq I$, and that its unique fixed point is a. If f is a solution of (2) $f''=f\circ g$ then $f''(a)=f\,[g(a)]=f(a)$ and

$$f'''=(f'\circ g)g', \tag{8}$$

so that $f'''(a)=f'\,[g(a)]\,g'(a)=-f'(a)$. Also,

$$\frac{1}{g'}f'''=f'\circ g$$

and

$$\left(\frac{1}{g'}f'''\right)'=(f''\circ g)g'=fg'$$

so that

$$\left(\frac{1}{g'}f'''\right)'-g'f=0.$$

This proves half of the following theorem.

THEOREM 3. *If g is a differentiable solution of $g\circ g=I$ and $g\neq I$, then the following are equivalent:*

$$f''=f\circ g \tag{c}$$

$$\begin{cases}\left(\dfrac{1}{g'}f'''\right)'-g'f=0\\ f''(a)=f(a)\\ f'''(a)=-f'(a),\end{cases} \tag{d}$$

where a is the unique fixed point of g.

It remains to be shown that (d) implies (c). We assume that f satisfies (d) and we define

$$H=f''-f\circ g.$$

As in Section 3 we will show that H satisfies a certain integral equation, but that this equation has only the zero solution.

We calculate

$$H'=f'''-(f'\circ g)g',$$

$$\frac{1}{g'}H'=\frac{1}{g'}f'''-f'\circ g,$$

and

$$\left(\frac{1}{g'}H'\right)'=\left(\frac{1}{g'}f'''\right)'-(f''\circ g)g'$$
$$=[f-f''\circ g]g'$$
$$=-(H\circ g)g'.$$

If we compose both sides with g, we obtain

$$H=-\left[\left(\frac{1}{g'}H'\right)'\circ g\right]g'$$

or

$$H=-\left[\left(\frac{1}{g'}H'\right)\circ g\right]'. \tag{9}$$

Now,

$$H'(a)=f'''(a)-f'[g(a)]g'(a)$$
$$=f'''(a)+f'(a)=0$$

so that if we integrate (9) we find

$$\int_{g(s)}^{a}H(t)dt=\left[\left(\frac{1}{g'}H'\right)\circ g\right]_a^{g(s)}$$
$$=H'(s)/g'(s)$$

or

$$H'(s)=g'(s)\int_{g(s)}^{a}H(t)dt. \tag{10}$$

Since $H(a)=f''(a)-f[g(a)]=f''(a)-f(a)=0$, another integration yields

$$H(x)=\int_a^x g'(s)\int_{g(s)}^{a}H(t)dtds.$$

If we interchange the order of integration this becomes

$$H(x)=\int_{g(x)}^{a}H(t)\int_{g(t)}^{x}g'(s)dsdt$$

or

$$H(x)=\int_{g(x)}^{a}[g(x)-t]H(t)dt. \tag{11}$$

This is the integral equation satisfied by H. As in Section 3 we iterate the integral equation, this time obtaining

$$H(x)=\int_{g(x)}^{a}\left\{\int_{g(t)}^{a}H(s)[g(t)-s]ds\right\}[g(x)-t]dt.$$

Now, an interchange in the order of integration yields

$$H(x) = \int_a^x H(s) \left\{ \int_{g(s)}^{g(x)} [g(t) - s][g(x) - t]dt \right\} ds. \tag{12}$$

The similarity between (6) and (12) is apparent; both are of the form $u(x) = \int_a^x u(s)\, K(s, x)\, ds$ where the kernel function K is continuous, and therefore bounded, as a function of s for any x. The argument of Section 3 then applies, and we see that H is the zero function. Thus $f'' - f \circ g = 0$ and the proof is complete.

The author is indebted to the reviewer for some helpful suggestions.

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ELEMENTARY TRANSCENDENTAL FUNCTIONS

G. P. SPECK, Bradley University

The term "transcendental" is employed in most introductory mathematics texts to describe the circular trigonometric, hyperbolic trigonometric, exponential, and logarithmic functions.

Occasionally in such texts a precise definition of "algebraic function" is given. A transcendental function can then be defined to be one which is not algebraic.

There are intermediate analysis books in which suggestions are given for proving that certain familiar functions, *defined for all real numbers*, are transcendental. However, it does not appear that such proofs for functions, *defined on any open interval*, are readily available in standard analysis texts.

In this paper we will establish that e^x, $\ln x$, the six basic circular trigonometric functions and their inverses, and the six basic hyperbolic trigonometric functions and their inverses are transcendental over any open interval on which they are defined.

DEFINITION 1. *A real (or complex) valued function f, defined and differentiable on the open interval (a, b), is algebraic over (a, b) iff there exist real (or complex) polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ with $P_n(x) \neq 0$ such that*

$$P_n(x)f^n(x) + P_{n-1}(x)f^{n-1}(x) + \cdots + P_1(x)f(x) + P_0(x)$$

is identically zero over (a, b).

Note. a may be $-\infty$ and/or b may be ∞. All functions considered in this paper will be taken to be defined and differentiable over (a, b). Ordinarily an irreducibility requirement is given in defining an algebraic function, but we do not choose to concern ourselves with this requirement.

DEFINITION 2. *A real (or complex) valued function, defined and differentiable over the interval (a, b), is transcendental over (a, b) iff it is not algebraic over (a, b).*

THEOREM 1. *If $f(x)=e^x$, where the domain of f is the open interval (a, b), then f is a transcendental function over (a, b).*

Proof. Suppose n is the least positive integer for which there exist polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ such that $P_n(x)\not\equiv 0$, degree of $P_0=\deg P_0=k$, and

$$P_n(x)e^{nx} + P_{n-1}(x)e^{(n-1)x} + \cdots + P_1(x)e^x + P_0(x) \equiv 0$$

over (a, b).

Now after $(k+1)$ successive derivatives are taken on both sides of the above identity the result is:

$$Q_n(x)e^{nx} + Q_{n-1}(x)e^{(n-1)x} + \cdots + Q_1(x)e^x \equiv 0$$

over (a, b) where $Q_n, Q_{n-1}, \cdots, Q_1$ are polynomials with $Q_n(x)\not\equiv 0$. Thus, $Q_n(x)e^{(n-1)x}+ \cdots +Q_2(x)e^x+Q_1(x)\equiv 0$ over (a, b), and we have a contradiction on the supposition that n is the least positive integer as required.

Therefore f is transcendental over (a, b).

THEOREM 2. *If $f(x)=\sinh x$, where the domain of f is the open interval (a, b), then f is a transcendental function over (a, b).*

Proof. Suppose $\sinh x=(e^x-e^{-x})/2$ is algebraic over (a, b) so that there exist polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ such that $P_n(x)\not\equiv 0$ and $\sum_{k=0}^{n} P_k(x)((e^x-e^{-x})/2)^k \equiv 0$ over (a, b). Now after expanding each expression of the form $((e^x-e^{-x})/2)^k$ and adding terms this identity becomes, after multiplying by e^{nx}: $\sum_{k=0}^{2n} Q_k(x)e^{kx} \equiv 0$ over (a, b), where $Q_{2n}, Q_{2n-1}, \cdots, Q_1, Q_0$ are polynomials with $Q_{2n}(x)\not\equiv 0$.

Thus e^x is transcendental over (a, b), a contradiction. Therefore our supposition is false, and f is transcendental over (a, b).

THEOREM 3. *If $f(x)=\cosh x$, where the domain of f is the open interval (a, b), then f is a transcendental function over (a, b).*

Proof. Consider $\cosh x=(e^x+e^{-x})/2$ and proceed as in the proof of Theorem 2.

THEOREM 4. *If f is transcendental over (a, b), then $1+f$ is transcendental over (a, b).*

Proof. If $1+f$ is algebraic over (a, b) then there exist polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ such that $P_n(x)\not\equiv 0$ and $\sum_{k=0}^{n} P_k(x)[1+f(x)]^k\equiv 0$ over (a, b). Now after expanding each expression of the form $[1+f(x)]^k$ and adding terms this identity becomes:

$$\sum_{k=0}^{n} Q_k(x)f^k(x) \equiv 0$$

over (a, b) where $Q_n, Q_{n-1}, \cdots, Q_1, Q_0$ are polynomials such that $Q_n(x) \not\equiv 0$.

Thus f is algebraic over (a, b), a contradiction. Hence, $1+f$ is transcendental over (a, b).

THEOREM 5. *If c is a nonzero constant, then f is transcendental over (a, b) iff cf is transcendental over (a, b).*

Proof. It is immediate that f is algebraic over (a, b) iff cf is algebraic over (a, b).

THEOREM 6. *If $f(x) \neq 0$ in (a, b), then f is algebraic over (a, b) iff $1/f$ is algebraic over (a, b).*

Proof. If f is algebraic over (a, b) then there exist polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ such that $P_n(x) \not\equiv 0$, $\sum_{k=0}^{n} P_k(x)f^k(x) \equiv 0$ over (a, b). Also, we can assume without loss of generality that $P_0(x) \not\equiv 0$. Thus, $\sum_{k=0}^{n} P_k(x)f^k(x) \equiv 0$ over (a, b) implies $\sum_{k=0}^{n} P_k(x)f^k(x)/f^n(x) \equiv 0$ over (a, b) implies $\sum_{k=0}^{n} P_k(x)[1/f(x)]^{n-k} \equiv 0$ over (a, b) implies $1/f$ is algebraic over (a, b).

The converse argument is similar.

THEOREM 7. *If $f(x) = \tanh x$, where the domain of f is the open interval (a, b), then f is transcendental over (a, b).*

Proof. Use $\tanh x = 1+(-2)/(1+e^{2x})$ and apply Theorems 4, 5, and 6 after seeing that e^{2x} is transcendental over (a, b).

THEOREM 8. *If f is defined on (a, b) and if $f^{-1} = g$ exists on (c, d) then f is algebraic over (a, b) iff $f^{-1} = g$ is algebraic over (c, d).*

Proof. If f is algebraic over (a, b) then there exist polynomials $P_n, P_{n-1}, \cdots, P_1, P_0$ such that $P_n(x) \not\equiv 0$ and $\sum_{k=0}^{n} P_k(x)f^k(x) \equiv 0$ over (a, b).

Let $y = f(x)$ so that $x = f^{-1}(y) = g(y)$. Then $\sum_{k=0}^{n} P_k(g(y))y^k \equiv 0$ over (c, d). Thus, expressing this identity in terms of powers of $g(y)$ on the left, we see that there exist polynomials $Q_m, Q_{m-1}, \cdots, Q_1, Q_0$ such that $Q_m(y) \not\equiv 0$ and $\sum_{k=0}^{m} Q_k(y)g^k(y) \equiv 0$ over (c, d).

Therefore, $f^{-1} = g$ is algebraic over (c, d).

The converse argument is similar.

Now using the results established to this point it is immediate that $\operatorname{csch} x = 1/\sinh x$, $\operatorname{sech} x = 1/\cosh x$, and $\coth x = 1/\tanh x$ are transcendental over any open interval on which they are defined. The same holds for each of the inverse hyperbolic trigonometric functions and for $\ln x$, the inverse of e^x.

Finally, e^{ix}, $\sin x = (e^{ix} - e^{-ix})/2i$, $\cos x = (e^{ix} + e^{-ix})/2$, and $\tan x = -i(1+(-2)/1+e^{2ix})$ can each be shown to be transcendental over an open interval (a, b) with proofs directly analogous to those in Theorems 1, 2, 3, and 7. It then follows by Theorems 6 and 8 that $\csc x$, $\sec x$, $\tan x$ and the six basic inverse circular trigonometric functions are transcendental over an open interval.

ANSWERS

A458. Define f such that $f(x)=0$ when x is algebraic and $f(x)=1$ when x is transcendental. f is discontinuous at each point, and since k and k^m are both algebraic or both transcendental, f has the desired property.

A459. In the first step, the definition of multiplication as repeated addition is valid only for integers. Hence, the function is not continuous and differentiation is meaningless for this definition.

A460. Since in the equation $r=\cos\theta/2$, θ may be replaced by $-\theta$ and the equation remains the same, the curve represented by the equation is symmetric with respect to the x axis. Next, r may be replaced by $-r$ and θ by $3\pi-\theta$ to yield a new equation for the same curve. The result is

$$-r = \cos(3\pi - \theta/2) = \cos 3\pi/2 \cos\theta/2 + \sin 3\pi/2 \sin\theta/2 = -\sin\theta/2$$

or $r=\sin\theta/2$. Hence the two curves are identical.

A 461.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2ab}{c^2} = \frac{2ab}{a^2+b^2} = e/g$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{b^2-a^2}{c^2} = \frac{b^2-a^2}{a^2+b^2} = b/g.$$

But $2ab$, a^2+b^2, and a^2-b^2 are generators of Pythagorean triples and the result follows.

A462. No. Suppose for contradiction that there exists such an f. Then there is an x_1 and x_2, $x_1\neq x_2$, such that $f(x_1)\neq f(x_2)$. Assume without loss of generality that $x_1<x_2$. Then there exists a $p'>0$ such that $x_1+p'=x_2$ and $f(x_1+p')=f(y)$ $=f(x)$, a contradiction.

(Quickies on page 225)

AN ALTERNATIVE TO THE GRAM-SCHMIDT PROCESS

JOHN H. STAIB, Drexel Institute of Technology

I dislike presenting the Gram-Schmidt process: it lacks elegance (in my opinion); it is not easy for the student to remember; and it is arithmetically so cumbersome that it makes testing of the student most difficult. Happily, while preparing for a lecture on the Gram-Schmidt process, I stumbled upon an alternative: a matrix method. If it is not original, it is at least not widely known.

We shall need some notation: E is an n-dimensional Euclidean space over the reals; $\mathcal{B}$ is an n by 1 column of independent vectors from E (thus, $\mathcal{B}$ represents an ordered basis for E); and $X_{\mathcal{B}}$ is an n by 1 column whose entries are the $\mathcal{B}$-coordinates of a vector X.

Inspiration for the method may be derived from the following theorem:

If A is an nth-order nonsingular real matrix, then $\mathcal{C}=A\mathcal{B}$ is also an ordered basis for E.

For this theorem prompts the following problem: *Given $\mathcal{B}$, find a nonsingular matrix A such that $\mathcal{C}=A\mathcal{B}$ is an orthogonal basis.*

A method for solving this problem is easily described: First, construct the matrix $B=[B^i\cdot B^j]_{nn}$, where B^i is the ith basis vector of $\mathcal{B}$. Then find a matrix A such that ABA' is a diagonal matrix, say D. This matrix A has the desired property.

The method is easily justified, requiring only some elementary theorems that usually appear in a first course in linear algebra. We begin by noting that the matrix B is simply the $\mathcal{B}$-coordinate matrix representation of E's dot product (inner product). Thus, we may write

$$X\cdot Y = X'_{\mathcal{B}} B Y_{\mathcal{B}}$$

Next, we introduce a new $\mathcal{C}$-coordinate system by the change-of-coordinate formula $Z_{\mathcal{B}}=A'Z_{\mathcal{C}}$. Then the $\mathcal{C}$-coordinate formula for E's dot product is given by

$$X\cdot Y = (A'X_{\mathcal{C}})'B(A'Y_{\mathcal{C}}) = X'_{\mathcal{C}}(ABA')Y_{\mathcal{C}} = X'_{\mathcal{C}}DY_{\mathcal{C}}.$$

But if the matrix representing E's dot product with respect to the $\mathcal{C}$-coordinate system is diagonal, then the basis $\mathcal{C}$ must be orthogonal! Now, a change-of-basis matrix is always the inverse of the transpose of the corresponding change-of-coordinates matrix. Thus, from

$$Z_{\mathcal{B}} = A'Z_{\mathcal{C}}$$

it follows that $\mathcal{B}=[(A')']^{-1}\mathcal{C}$ or, $\mathcal{C}=A\mathcal{B}$.

Example. Let E be the space of polynomials of the form at^3+bt^2+ct+d, where the dot product is defined by

$$p\cdot q = \int_{-1}^{1} p(t)q(t)dt.$$

Find an orthogonal basis for E.

SOLUTION:

1. Make any convenient choice of a basis for E, say $\mathcal{B}=[1, t, t^2, t^3]'$.
2. Construct the matrix $B=[B^i\cdot B^j]_{44}$.

$$\begin{bmatrix} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \\ 0 & 2/5 & 0 & 2/7 \end{bmatrix}.$$

3. We now seek an A such that ABA' is diagonal. Since A need not be orthogonal, we need not solve the associated eigenvalue problem; rather, we use a scheme similar to the one frequently used to find the inverse of a matrix:

B				I			
2	0	2/3	0	1	0	0	0
0	2/3	0	2/5	0	1	0	0
2/3	0	2/5	0	0	0	1	0
0	2/5	0	2/7	0	0	0	1
2	0	2/3	0	1	0	0	0
0	2	0	6/5	0	3	0	0
2	0	6/5	0	0	0	3	0
0	2	0	10/7	0	0	0	5
2	0	2/3	0	1	0	0	0
0	2	0	6/5	0	3	0	0
0	0	8/15	0	−1	0	3	0
0	0	0	8/35	0	−3	0	5

T (upper triangular) A (lower triangular)

That is, we use row operations to reduce B to upper triangular form, while at the same time transforming I into lower triangular form. (This is possible when B is positive definite, which is the case when B represents a dot product.) That the resulting lower triangular matrix A fills the desired role is easily seen:

$$ABA' = TA' = (\text{upper})(\text{upper}) = (\text{upper}).$$

But

$$ABA' \text{ is symmetric! (Why?)}$$

4. Finally, we take

$$\mathcal{C} = A\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & -3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3t \\ 3t^2 - 1 \\ 5t^2 - 3 \end{bmatrix}.$$

Note. In finding an A such that ABA' is diagonal, we are in no way concerned about the nature of the diagonal entries. Thus, row multiplication can be freely used to avoid fractions. This means that we can devise computer programs that will not suffer from round-off error.

EUCLIDEAN CONSTRUCTIBILITY IN GRAPH-MINIMIZATION PROBLEMS

E. J. COCKAYNE, University of Victoria and Z. A. MELZAK, University of British Columbia

1. Let $b_1, \cdots, b_N$ be any set of distinct points in the plane. By a tree U on the vertices $b_1, \cdots, b_N$ we mean any set consisting of some of the $\binom{N}{2}$ closed straight segments b_ib_j with the property that any two vertices can be joined by a sequence of segments belonging to U in one and only one way. A segment b_ib_j is called a branch of U, the length $L(U)$ of U is the sum of the lengths of its branches, $\{b_i\}$ is the set of all vertices sending branches to the vertex b_i and $w(b_i)$ is their number. We now formulate the problem:

($S_{n\alpha\beta\gamma}$): *Given three nonnegative real numbers α, β, γ and n distinct points $a_1, \cdots, a_n$ in the plane, to find an integer k and k additional points $s_1, \cdots, s_k$, and to construct the tree(s) U on the vertices $a_1, \cdots, a_n, s_1, \cdots, s_k$ so as to minimize the sum*

$$T = L(U) + \alpha \sum_{i=1}^{n} w(a_i) + \beta \sum_{i=1}^{k} w(s_i) + \gamma k.$$

We offer now an economic interpretation of the problem ($S_{n\alpha\beta\gamma}$). Let the points $a_1, \cdots, a_n$ represent n cities and let the tree U represent a system of roads connecting the cities. Let a point at which s roads meet, $s \geqq 3$, be called an s-junction. Suppose that the cost of building one unit of length of road is 1 (in some monetary units), that a city s-junction costs $s\alpha$, any other s-junction costs $s\beta$, and in addition there is a fixed charge γ for each junction outside of a city. Now ($S_{n\alpha\beta\gamma}$) is formally identical with asking: What is the cheapest system of roads that connects the n cities?

If $\alpha = \beta = \gamma = 0$, ($S_{n\alpha\beta\gamma}$) reduces to the classical Steiner problem (S_n): Given n distinct points $a_1, \cdots, a_n$ in the plane, to construct the shortest tree(s) whose vertices include $a_1, \cdots, a_n$ and any set of k additional points $s_1, \cdots, s_k (k \geqq 0)$.

Now suppose $\beta = 0$ and $\alpha > \gamma$ where γ is sufficiently large. Then T will be smallest when each $w(a_i)$ has its minimum value 1 and as few extra vertices as possible are adjoined. Since trees are connected, $w(a_i) = 1$ for each i implies that $k \geqq 1$. Therefore, when $\beta = 0$ and for suitable α, γ, the minimizing trees of ($S_{n\alpha\beta\gamma}$) will be precisely the minimum length trees among those having $w(a_i) = 1$ for each i and $k = 1$, i.e., ($S_{n\alpha\beta\gamma}$) reduces to

(P_n): Given n distinct points $a_1, \cdots, a_n$ in the plane to find the point p which minimizes the function $\sum_{i=1}^{n} pa_i$.

In [2] and [3] we have proved the existence of solutions of these problems and have given an algorithm for the solution of (S_n) by a finite sequence of Euclidean constructions (i.e., ruler-compass constructions in the classical sense). The purpose of this note is to show that no such algorithm exists for ($S_{n\alpha\beta\gamma}$). More precisely, we prove

THEOREM. (P_n), *(and hence ($S_{n\alpha\beta\gamma}$)) is not, in general, solvable by Euclidean constructions.*

2. We use $n=5$ for our example. (P_3) is solvable by Euclidean constructions and the solution of (P_4) is the intersection of the diagonals if the configuration is convex and the vertex interior to the convex hull otherwise. We take 5 points $A_i (i=1, \cdots, 5)$ symmetrically placed with respect to the x-axis as shown in Figure 1.

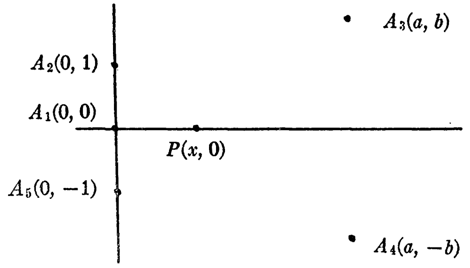

FIG. 1.

The minimum point P lies on the x-axis with its co-ordinate in $[0, a]$.

$$\sum_{i=1}^{5} PA_i = x + 2\sqrt{1 + x^2} + 2\sqrt{b^2 + (a - x)^2}.$$

Minimizing this function by elementary methods, we find that the coordinate x of P satisfies an eighth degree polynomial equation $f(x)=0$ whose coefficients are polynomials in a and b.

We show that for suitable integers a, b, $f(x)=0$ has Galois group over the rationals which does not have order 2^k where k is a positive integer. Therefore x is not an element belonging to an extension field of the rationals of degree 2^k and hence the segment OP is not constructible by Euclidean constructions (see [4] page 185); i.e., for suitable choices of the five points (P_5) is not solvable by Euclidean constructions.

The leading coefficient of $f(x)$ is 15. In order to use the theory of [4] page 190–191, we need to work with a monic polynomial and therefore make the transformation $x=y/15$ and multiply the equation through by 15^7, thus obtaining equation $g(y)=0$ where $g(y)$ is monic. We note that such a transformation does not affect reducibility over the rationals or the Galois group of the equation. The coefficients of $g(y)$ are:

$$\begin{aligned}
y^8:&\quad 1\\
y^7:&\quad -60a\\
y^6:&\quad 15(90a^2 + 22b^2 + 22)\\
y^5:&\quad -15^2(88a + 60a^3 + 44ab^2)\\
y^4:&\quad 15^3(15a^4 + 22a^2b^2 - 9b^4 + 132a^2 + 60b^2 - 9)\\
y^3:&\quad 15^4(88a^3 + 120ab^2 - 36a)\\
y^2:&\quad 15^5(22a^4 + 60a^2b^2 + 6b^4 + 6b^2 - 54a^2)\\
y:&\quad 15^6 \cdot 12a \cdot (3a^2 - b^2)\\
1:&\quad -15^7(b^2 - 3a^2)^2.
\end{aligned}$$

For $a=b=3$, $g(y)$ reduces mod 11 to

$$\overline{g(y)} = y^8 + 7y^7 + 6y^6 + 7y^5 + y^4 + 9y^3 + 8y^2 + 7y + 8$$

which has the irreducible factorization mod 11

$$(y^3 + 10y^2 + 7y + 6)(y^5 + 8y^4 + 7y^3 + 7y^2 + 10y + 5).$$

We deduce that over the rationals $g(y)$ is either irreducible or factors into an irreducible cubic and an irreducible quintic. These respective factors and the result of [4] p. 120 show that neither $g(y)$ nor $\overline{g(y)}$ has a multiple root. Thus, using the theory of [4] pp. 190–191, the Galois group of $g(y)=0$ over the rationals contains a permutation whose representation in disjoint cycles consists of exactly two cycles of lengths 3 and 5 and whose order is therefore 15. The order of this Galois group is a multiple of 15 and hence not a power of 2. This completes the proof of the theorem.

Primes 2, 3, 5 could not be used since, in each of these cases, the reduced polynomial has multiple roots for any a, b. The University of Victoria's IBM 360 Model 44 machine was used to factorize $g(y)$ into irreducible factors first mod 7 and then mod 11 for a series of values of a and b. The prime 11 with $a=b=3$ produced the required counterexample.

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ON THE REVERSING OF DIGITS

LEONARD F. KLOSINSKI and DENNIS C. SMOLARSKI, University of Santa Clara

A short time ago the following article appeared in the newspaper of a small western city:

A mathematics professor complained to the policeman that a student had almost run him down as he attempted to cross the street.

"Did you get his license number?" asked the policeman.

"Not exactly," the professor said. "But I do remember that if the number was doubled and then multiplied by itself, the square root of the product was the original number with the integers reversed."

The problem suggested is simply this: what number m when multiplied by two results in the original number with the digits written in reverse order? A few minutes of thought or several hours on the digital computer reveals the answer, or at least leads one to make a conjecture. There is the trivial solution $m=0$, but there is no other solution. This is readily seen when one considers a general n digit number which we will write as $abc \cdots xyz$. Upon multiplication by 2, an $n+1$ digit number will be obtained if a is greater than 4, and hence a can equal only 1, 2, 3, or 4. (Clearly, a cannot equal 0 or we really do not have an n digit number.) Furthermore, multiplying $abc \cdots xyz$ by 2 results in an even number; since the resulting number is to be $zyx \cdots cba$, a must be even and hence 2 or 4. If $a=2$, then z must be 1 or 6. Now note that $2a$ plus a possible carry must be 1 or 6. Since the carry will be at most 1, one sees that $2 \cdot 2$ plus carry cannot equal 1 or 6. A similar argument disposes of the case $a=4$ and thereby completes the solution to the problem, i.e., there is no number m, other than 0 for which doubling reverses the digits.

A generalization of the original problem leads to the question: Is there any integer $m>0$ which when multiplied by an integer n, $n>2$ (n must be a one digit number, of course), results in the original number with the digits reversed? A more satisfactory solution is found to this question since one quickly discovers that 219978021780219978 when multiplied by 4 reverses the digits. There are in fact an unlimited supply of numbers which when multiplied by 4 produce the original number with the digits reversed. The two smallest numbers with this property are 2178 and 21978. By an argument similar to that employed above, one determines that if $4m$ is equal to m with the digits reversed, then m is some combination of the three numbers 2178, 0, and $21999 \cdots 978$ (the number of 9's in this last number is arbitrary).

One also observes a similar property holds for numbers m when multiplied by $n=9$. Here the smallest number m is 1089 with the next larger being 10989. Here again one can display an infinite number of integers m with the desired property. While the supply of numbers $m>0$ is without limit, there are only two numbers $n>1$, $n=4$ and $n=9$, which can reverse the digits of m.

Let us generalize the original problem one step further: Do there exist other bases k in which the multiplication of an integer $m>0$ by an integer $n>1$ results in m with digits reversed? We will now demonstrate how to find numbers m and n.

Let k be the base under consideration and let t be an integer such that $0<t\leqq k-1$. Consider the number $(k-t)/t$; if $(k-t)/t$ is an integer greater than one then let $n=(k-t)/t$ and m equal the number whose first digit is t, second digit is $t-1$, third digit is $k-t-1$, and fourth digit is $k-t$. Then we claim that

$$((k-t)/t)[t(t-1)(k-t-1)(k-t)] = (k-t)(k-t-1)(t-1)t.$$

This is easily proven by writing m as $t\cdot k^3+(t-1)k^2+(k-t-1)k+(k-t)$, and m with digits reversed as $(k-t)k^3+(k-t-1)k^2+(t-1)k+t$.

Algebraic manipulation then gives

$$\begin{aligned}((k-t)/t)[t\cdot k^3 + (t-1)k^2 + (k-t-1)k + (k-t)]\\ &= k^4 + k^3 - k^2 - k - tk^3 - tk^2 + tk + t\\ &= (k-t)k^3 + (k-t-1)k^2 + (t-1)k + t.\end{aligned}$$

It is not claimed that these are the only numbers m and n which produce the desired result, but simply that the numbers given will work. Other numbers can be found; for example, if the base is of the form $3k+2$, then the number with first digit k and second digit $2k+1$ when multiplied by 2 will reverse the digits.

We conclude this discussion by replying to one more question. In what base, which number when converted to base 10 gives the smallest number that can be multiplied by 2 and cause the reversal of the original number? We claim that this number, written in base 5, is 13. The next time you see an automobile with license plate number 8, remember that you can multiply by 2 in base 5 and obtain 31, the reverse of 13, which is eight in base 5.

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ON THE TRINOMIAL COEFFICIENTS

R. L. KEENEY, Massachusetts Institute of Technology

The binomial expansion is $(x+y)^N = \sum_{n=0}^{N} c(n, N)x^n y^{N-n}$, where the binomial coefficients $c(n, N)$ may be calculated directly from their definition $c(n, N) = N!/n!\,(N-n)!$. Using Pascal's triangle, one has a geometric interpretation of an iterative method for calculating these coefficients for any particular N. This is illustrated in Figure 1. The value of $c(n, N)$ is calculated by adding $c(n, N-1)$ and $c(n-1, N-1)$ found directly above $c(n, N)$ in the triangle. Algebraically stated,

$$c(n, N) = c(n-1, N-1) + c(n, N-1), \qquad \text{for } N \geqq 1.$$

The term $c(0, 0)=1$, and $c(n, N)=0$ for $N<0$ or $n>N$ by definition.

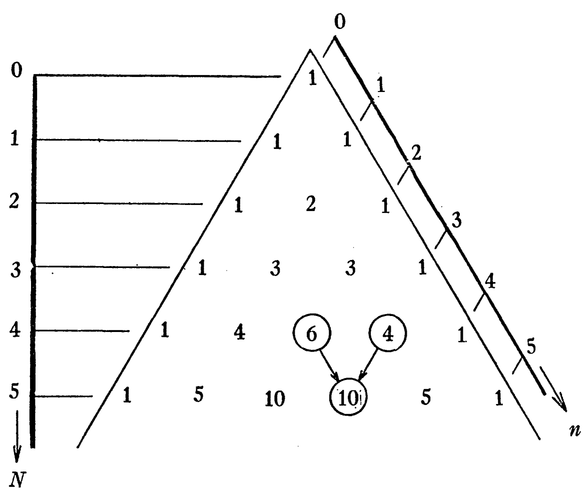

FIG. 1

One can construct a pyramid of the trinomial coefficients in a manner similar to the way Pascal's triangle is constructed for the binomal coefficients. Analogous to the binomial case, the trinomial expansion is

$$(x+y+z)^N = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} c(n_1, n_2, N) x^{n_1} y^{n_2} z^{N-n_1-n_2}$$

where the trinomial coefficients $c(n_1, n_2, N) = N!/n_1!\, n_2!\, (N-n_1-n_2)!$. The value of $c(n_1, n_2, N)$ can be calculated by adding $c(n_1-1, n_2, N-1)$, $c(n_1, n_2-1, N-1)$, and $c(n_1, n_2, N-1)$, which are the three coefficients found above $c(n_1, n_2, N)$ in Figure 2. Algebraically, $c(n_1, n_2, N) = c(n_1-1, n_2, N-1) + c(n_1, n_2-1, N-1) + c(n_1, n_2, N-1)$, where $c(n_1, n_2, N)$ is defined as zero when any of the following are true: $N<0$, $n_1<0$, $n_2<0$, or $n_1+n_2>N$. Also, $c(0, 0, 0)=1$ by definition.

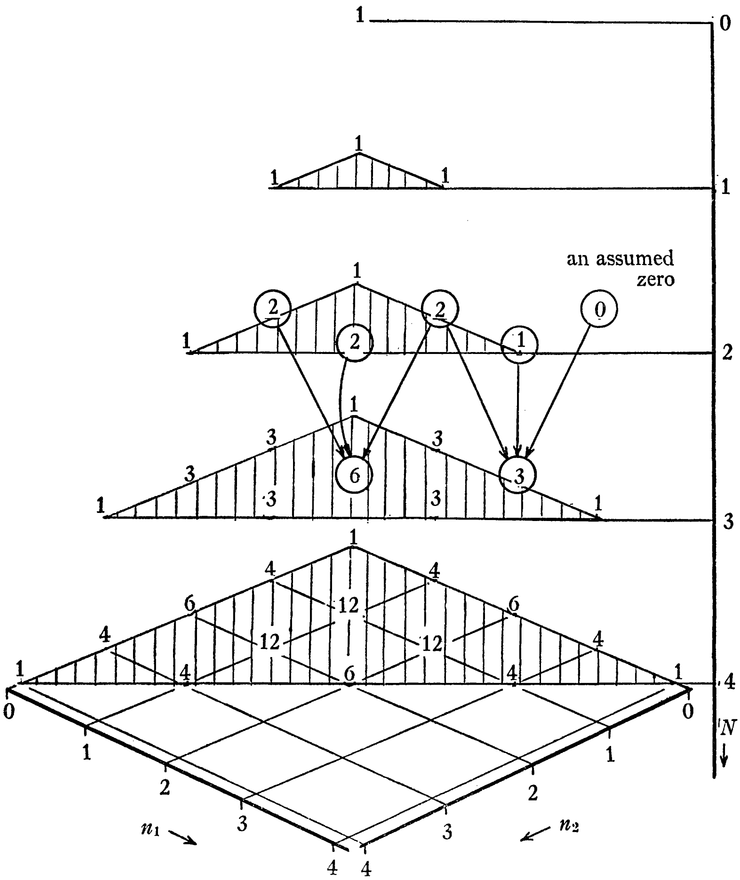

FIG. 2

To obtain $c(1, 1, 3)$, we add the three terms in the $N=2$ plane directly above the $c(1, 1, 3)$ position. As seen in the figure, these terms are $c(0, 1, 2)$, $c(1, 0, 2)$, and $c(1, 1, 2)$ which all equal 2. Therefore, $c(1, 1, 3) = 2+2+2 = 6$. Similarly, to calculate $c(2, 0, 3)$, we add the three terms in $N=2$ plane above the $c(2, 0, 3)$ position. These terms are $c(1, 0, 2)$, $c(2, -1, 2)$, and $c(2, 0, 2)$ which equal 2, 0, and 1, respectively. Therefore, $c(2, 0, 3)=3$.

In Figure 3, we indicate the plane from the pyramid corresponding to $N=4$. Thus, when expanding $(x+y+z)^4$, the coefficient of the x^2yz term, $c(2, 1, 4)$ is 12, and the coefficient of the x^0yz^3 term, $c(0, 1, 4)$, is 4.

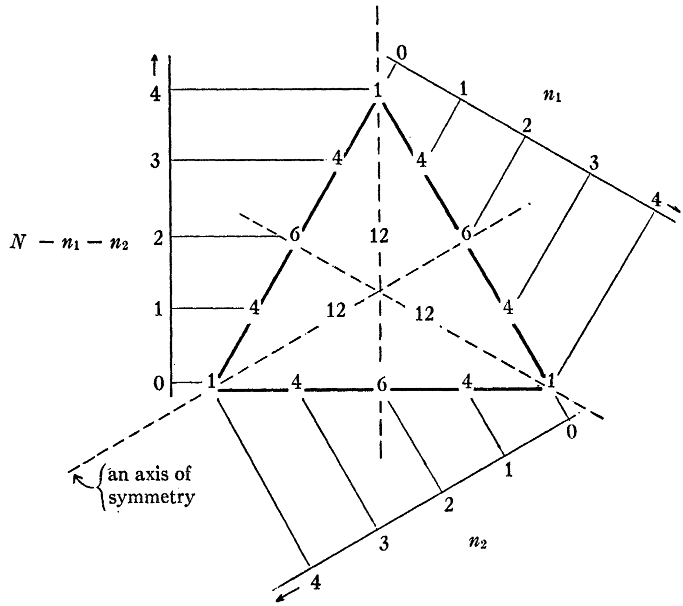

FIG. 3

As indicated in Figure 3, there are three axes of symmetry in the plane of coefficients corresponding to any N.

One should also note that when either n_1, n_2, or $N-n_1-n_2$ is zero, the trinomial coefficients are just the binomial coefficients corresponding to the respective N.

As a physical explanation of the construction of the pyramid, consider an urn with red, black, and white balls. We draw a ball, note its color, and replace it N different times. The coefficient $c(n_1, n_2, N)$ may be interpreted as the number of permutations in which we can draw n_1 red balls, n_2 black balls, and $N-n_1-n_2$ white balls. There are only three ways to do this. We can draw n_1-1 red balls and n_2 black balls in the first $N-1$ trials and then draw a red ball, or we can draw n_1 red balls and n_2-1 black balls in the first $N-1$ trials and then draw a black ball, or we can first draw n_1 red balls and n_2 black balls in $N-1$ trials and then draw a white ball on the last trial.

Obviously, this discussion could be extended to cover the multinomial coefficients, but little insight would be gained by this. However, let us mention that the multinomial coefficients of $(x+y+z+w)^N$ form a pyramid for *each* N, and that these coefficients are evaluated by summing four coefficients in the pyramid corresponding to $N-1$.

I nominate the following as the proof of the month:

"Candide . . . was very much grieved at having to part with his sheep, which he left with the Academy of Sciences at Bordeaux. The Academy offered as the subject for a prize that year the cause of the redness of the sheep's fleece; and the prize was awarded to a learned man in the North, who proved by A plus B minus C divided by Z that the sheep must be red and die of the sheep-pox."—*Candide*, Chap. 22.

JAMES F. RAMALEY

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before December 1, 1969

PROPOSALS

733. *Proposed by Richard L. Breisch, University of Colorado.*

Solve this determinant equation with nonnegative digits: $(R\neq 0)$

$$R\ A\ T\ S = -\begin{vmatrix} E & A & T \\ A & T & E \\ T & E & A \end{vmatrix}.$$

734. *Proposed by Simeon Reich, Israel Institute of Technology, Haifa, Israel.*

Let K be a plane convex curve which can rotate in an equilateral triangle of altitude h so as to be always tangent to its three sides. Prove that:

a) If $h>2\pi/(2\pi-3\sqrt{3})$, then K always encloses a lattice point.

b) If $h<1+\sqrt{3}/2$, then one can rotate and translate the curve K in the plane so that in one position at least it will contain no lattice points.

735. *Proposed by Charles W. Trigg, San Diego, California.*

Find a triangular number which can be partitioned into three 3-digit primes which together contain the nine positive digits.

736. *Proposed by Alfred Kohler, Long Island University.*

Al, Bill, Chuck, and Don all live in the same school district. When Al faces the school from his home, Bill's home is directly to Al's right. Bill lives directly to the west of school, and Chuck lives due south of Bill. At his home, Don can see the sun setting behind Chuck's house at times.

Al's home is as far from Bill's as it is from school, while Chuck lives twice as far from Bill as Bill does from school. Don lives three times as far from Chuck as Chuck does from school, and Don lives ten times as far from school as does Al.

Show that Al's home, Don's home, and the school are collinear.

737. *Proposed by Irving A. Dodes, Kingsborough Community College, Brooklyn, New York.*

Mr. S. D. S. Hippie, inveterate seeker after truth (far after), awoke from a slight doze to hear his plane geometry teacher remark that if the midpoints of any quadrilateral are joined, the result is a parallelogram. Not to be outdone in untenable hypotheses, Mr. Hippie opined that if trisection points are joined, the result will also be a parallelogram.

Prove that no other dividing points independent of the lengths of the sides exist to produce a parallelogram.

738. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

There is a river with parallel and straight shores. A is located on one shore and B on the other, with $AB=72$ miles. A ferry boat travels the straight path AB from A to B in four hours and from B to A in nine hours. If the boat's speed on still water is $v=13$ mph, what is the velocity of the flow?

739. *Proposed by Mehmet Agargun, Diyarbakir, Turkey.*

Let the vertical projections of a variable point P of a central conic with center O, on the major and minor axes be C and S, and the tangent line at P intersects these axes at C' and S' respectively. Then prove that:

$$\overline{OC}\cdot\overline{OC'} - \overline{OS}\cdot\overline{OS'} = \text{constant}.$$

SOLUTIONS

Late Solutions

C. R. Berndtson, MIT Lincoln Laboratory: **710**; *Darel W. Hardy, Colorado State University:* **711**; *Thomas Schewczyk, University of Wisconsin at Waukesha:* **711**; *Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan:* **708, 711**; *Bill Knight, Laramie, Wyoming:* **697**; *Michael J. Martino, Temple University:* **692**; *Simeon Reich, Israel Institute of Technology, Haifa, Israel:* **706, 708, 711**; *Charles W. Trigg:* **699, 700**; *Bryant Tuckerman, IBM Research Center, Yorktown Heights, New York:* **691.**

Errata

The name of *Michael J. Martino*, the proposer, was omitted from the solvers of Problem 691, January, 1969, page 46.

The last line of the solution to Problem 695, January, 1969, page 48, should read:

$$a(b!)(x-1)!/(x-a)!(a+b)!$$

The name of the proposer of Problem 707, p. 283, November, 1968, should read Joseph Malkevitch.

A Product Cryptarithm

712. [*January*, 1969] *Proposed by C. R. J. Singleton, Petersham, Surrey, England.*

A four-digit number is equal to the product of three numbers of one, two, and three digits, respectively. The ten digits of these four numbers are all different. Find the two solutions.

Solution by Otto Mond, White Plains, New York.

Consider the ten digits to be represented by the letters A through J. Then we have

$$ABCD = E \cdot FG \cdot HIJ.$$

From the fact that we have only a four-digit product, it is obvious that E must be either 1 or 2. Similarly, the letters F and H, the high-order digits, must be 1, 2 or 3. Trying these combinations while noting that the units digits cannot be zero without causing a repeated digit, we find:

$$8970 = 1 \times 26 \times 345$$

and

$$8596 = 2 \times 14 \times 307.$$

Also solved by Donald Batman, Parsippany, New Jersey; Richard L. Breisch (partially), University of Colorado; Harry M. Gehman, SUNY at Buffalo, New York; Philip Haverstick, Fort Belvoir, Virginia; William Jager, Oakland, California; Prasert Na Nagara, Kasetsart University, Bangkok, Thailand; Richard D. Nation, Jr., San Diego, California; G. A. Novacky, Jr., Wheeling College, West Virginia; Patricia M. Osmer, University of Bridgeport, Connecticut; Edward F. Schmeichel, College of Wooster, Ohio; E. P. Starke, Plainfield, New Jersey; Paul Sugarman, Massachusetts Institute of Technology; Charles W. Trigg, San Diego, California; Chrisopher P. Urbanski, Montreal, Canada; and the proposer.

Walter Penny, Greenbelt, Maryland, found the problem in *Table Talk*, Vol. 7, No. 9, July, 1961.

The Smart or Dumb Proctor

713. [January, 1969] *Proposed by Brother Alfred Brousseau, St. Mary's College, California.*

An examination was prepared as follows. Thirty-six questions were made and each assigned a number corresponding to the figures that might come up using two differentiated dice. Contestants were given questions by throwing the two dice. If the number that came up belonged to a question already used, the throw was repeated. What would be the expected number of throws needed to select all thirty-six questions?

Solution by Donald Batman, Parsippany, New Jersey.

Assume that n questions have already been selected, so that the probability is $n/36$ that a throw of the dice will not yield another question. The probability that it will take exactly m throws to select a question in this situation is:

$$\phi_n(m) = \left(\frac{n}{36}\right)^{m-1} \cdot \frac{36 - n}{36} \cdot$$

The expected value of the random variable m is

$$E_n(m) = \sum_{m=1}^{\infty} m\cdot\phi_n(m) = \sum_{m=1}^{\infty}\left[m\cdot\left(\frac{n}{36}\right)^{m-1} - m\left(\frac{n}{36}\right)^{m}\right]$$
$$= \sum_{k=0}^{\infty}(k+1)\left(\frac{n}{36}\right)^{k} - \sum_{k=1}^{\infty} k\left(\frac{n}{36}\right)^{k} = 1 + \sum_{k=1}^{\infty}\left(\frac{n}{36}\right)^{k} = \left(1-\frac{n}{36}\right)^{-1}.$$

At this point, there are two solutions to the problem, depending upon whether the person conducting the examination persists in throwing the dice to select the last outstanding question, hence: Smart Proctor (selects 35 questions):

$$\text{Total throws} = \sum_{k=0}^{34} E_k = \sum_{k=0}^{34}\left(1-\frac{k}{36}\right)^{-1}$$
$$= 36\sum_{k=0}^{34}\frac{1}{36-k} = 36\sum_{k=2}^{36}\frac{1}{k} \cong 114.$$

Dumb Proctor (selects 36 questions):

$$\text{Total throws} = \sum_{k=0}^{35} E_k = 36\sum_{k=1}^{36}\frac{1}{k} \cong 150.$$

Also solved by Steven A. Baril, Michigan Technological University; Gerard E. Dallal, Brooklyn, New York; W. W. Funkenbusch, Michigan Technological University; Michael Goldberg, Washington, D.C.; John E. Hafstrom, California State College at San Bernardino; James C. Hickman, University of Iowa; John M. Howell, Los Angeles City College; Arthur Ketterer, Hillside, New Jersey; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Lew Kowarski, Morgan State College, Maryland; J. R. Kuttler and N. Rubinstein, Johns Hopkins University, Applied Physics Laboratory; Edward F. Schmeichel, College of Wooster, Ohio; C. P. Urbanski, Montreal, Canada; and the proposer. Two incorrect solutions were received.

References were found by Goldberg: *Probability and Frequency*, by H. C. Plummer, Macmillan, London, 1940, pp. 39–40; Hickman: *Choice and Chance*, by Whitworth, Exercise 685; Howell: *Fifty Challenging Problems of Probability*, by Mosteller; and, Klamkin: *Introduction to Probability Theory and Its Application*, by Feller, and *The double dixie cup problem*, American Mathematical Monthly, January, 1960, pp. 58–61.

Twelve-Tone Scale?

714. [January, 1969] *Proposed by Samuel Wolf, Linthicum Heights, Maryland.*

Find a solution in base 12 for

$$\begin{array}{rrrrrrr} & V & I & O & L & I & N \\ & V & I & O & L & I & N \\ & & & V & I & O & L & A \\ & & & C & E & L & L & O \\ \hline Q & U & A & R & T & E & T \end{array}$$

Solution by Nigel F. Nettheim, U. S. Department of Commerce, Washington, D. C.

I have regarded this as a problem in computer programming. There are 8 solutions in base 12 and they are given below. Computer time on a GE625 was 46 seconds.

	V	*I*	*O*	*L*	*I*	*N*
	V	*I*	*O*	*L*	*I*	*N*
		V	*I*	*O*	*L*	*A*
		C	*E*	*L*	*L*	*O*
Q	*U*	*A*	*R*	*T*	*E*	*T*
	11	4	3	6	4	1
	11	4	3	6	4	1
		11	4	3	6	5
		9	8	6	6	3
2	0	5	7	10	8	10
	8	3	7	5	3	2
	8	3	7	5	3	2
		8	3	7	5	0
		9	4	5	5	7
1	6	0	10	11	4	11
	9	0	4	10	0	2
	9	0	4	10	0	2
		9	0	4	10	3
		5	8	10	10	4
1	7	3	6	11	8	11
	9	5	4	10	5	3
	9	5	4	10	5	3
		9	5	4	10	2
		6	7	10	10	4
1	8	2	11	0	7	0
	5	9	8	3	9	6
	5	9	8	3	9	6
		5	9	8	3	11
		10	2	3	3	8
1	0	11	4	7	2	7
	6	8	0	3	8	9
	6	8	0	3	8	9
		6	8	0	3	4
		5	11	3	3	0
1	2	4	7	10	11	10
	11	7	8	5	7	10
	11	7	8	5	7	10
		11	7	8	5	9
		6	3	5	5	8
2	0	9	4	1	3	1

	6	7	10	8	7	11
	6	7	10	8	7	11
		6	7	10	8	4
		5	9	8	8	10
1	2	4	3	0	9	0

There are 40 solutions in base 13, for example:

	11	3	6	7	3	0
	11	3	6	7	3	0
		11	3	6	7	9
		4	8	7	7	6
1	10	9	12	2	8	2

There are many solutions in higher bases.

Also solved by Andrew N. Aheart, West Virginia State College; Michael G. Jacobs, Cypress Junior College, California; Prasert Na Nagara, Kasetsart University, Bangkok, Thailand; C. P. Urbanski, Montreal, Canada; and the proposer.

Magic Matrix

715. [January, 1969] *Proposed by John Brillart, Berkeley, California.*

Given the classical 3×3 magic square

$$M = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$$

Compute M^n for $n \geqq 2$.

I. *Solution by E. P. Starke, Plainfield, New Jersey.*

Let I be the identity matrix and U be the matrix in which every element is 1. Ordinary multiplication of matrices gives

$$M^2 = 67\cdot U + 24\cdot I, \qquad MU = 15\cdot U.$$

Then an easy induction establishes the result in the form

$$M^{2k+1} = 67\left(\sum_{j=1}^{k} 15^{2k-2j+1}\cdot 24^{j-1}\right)\cdot U + 24^k\cdot M,$$

$$M^{2k} = 67\left(\sum_{j=1}^{k} 15^{2k-2j}\cdot 24^{j-1}\right)\cdot U + 24^k\cdot I,$$

with $k=1, 2, \cdots$

II. *Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

The problem here is a special case of the standard one of calculating the matrix functional value $F(M)$ where $F(z)=z^n$. As is well known (F. R. Gantmacher, *Theory of Matrices*, Chelsea, New York, 1959), $F(M)=L(M)$ where $L(z)$ is the Lagrange-Hermite polynomial "interpolating" $F(z)$ at the characteristic values $\lambda_1, \lambda_2, \lambda_3$ of M. Here

$$L(z) = \frac{(z-\lambda_2)(z-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}F(\lambda_1) + \frac{(z-\lambda_3)(z-\lambda_1)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}F(\lambda_2) + \frac{(z-\lambda_1)(z-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}F(\lambda_3)$$

if $\lambda_1 \neq {}_2\lambda \neq \lambda_3$. (The cases for equality of eigenvalues can be gotten by using L'Hospital's rule.)

Since the sum of each row of the matrix $=15$, one eigenvalue is 15. It is then easy to show that the other two eigenvalues satisfy $\lambda^2=24$. Thus

$$M^n = \frac{M^2-24I}{221}(15)^n + \frac{M^2-(15-2\sqrt{6})M-30I\sqrt{6}}{48-60\sqrt{6}}(2\sqrt{6})^n + \frac{M^2-(15+2\sqrt{6})M+30I\sqrt{6}}{48+60\sqrt{6}}(-2\sqrt{6})^n.$$

For several other methods for finding M^n, see Problem 64-19, *The Nth power of a matrix*, SIAM Review, April, 1966, pp. 238–239.

Also solved by J. C. Binz, Bern, Switzerland; Richard L. Breisch, University of Colorado; Michael Goldberg, Washington, D.C.; Arnold Hammel, Mt. Pleasant, Michigan; Philip Haverstick, Ft. Belvoir, Virginia; Peter A. Lindstrom, Genesee Community College, New York; Michael J. Martino, Temple University; Francis D. Parker, St. Lawrence University; Simeon Reich, Israel Institute of Technology, Haifa, Israel; Harry M. Rosenblatt, U. S. Bureau of the Census, Washington, D.C.; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Bucknell University; C. P. Urbanski, Montreal, Canada; and the proposer.

Matching Draws

716. [January, 1969] *Proposed by John M. Howell, Los Angeles City College.*

Suppose that we have a deck of cards numbered $1, 2, \cdots n$ and a second deck numbered $1, 2, \cdots (n+d)$, $d \geqq 0$, and that we shuffle the two decks and draw one card at a time from each until the smaller deck is exhausted. Find the probability of r matching draws and the mean and variance of the number of matching draws.

Solution by Michael Goldberg, Washington, D. C.

Problems of this type were considered by Montmort and DeMoivre. Their results are given on Pages 91–93 and 153–155 of Todhunter's *History of the Theory of Probability*, 1865.

The special case for which $d=0$ is given by H. C. Plummer in *Probability and Frequency*, 1940, Pages 20–21. The probability $u_{n,r}$ of r matching draws is given by

$$u_{n,r} = n!\{1/2! - 1/3! + 1/4! - \cdots + (-1)^{n-r}/(n-r)!\}/r!$$

Hence, the mean number r_m of matching draws is $r_m=\sum_{r=1}^{n} ru_{n,r}$, and the variance is $\{\sum_{r=1}^{n}(r-r_m)^2\}/(n-1)$.

Also solved by E. F. Schmeichel, College of Wooster, Ohio; and the proposer.

A Pentagon Property

717. [January, 1969] *Proposed by V. F. Ivanoff, San Carlos, California.*

In a convex pentagon we are given the areas of five triangles each formed with two sides and one diagonal. Find the areas of the remaining five triangles each formed with one side and two diagonals.

Solution by E. P. Starke, Plainfield, New Jersey.

Let the vertices of the pentagon, referred to rectangular axes, be $A(0, 0)$, $B(a, 0)$, $C(b, c)$, $D(d, e)$, $E(f, g)$, where $a, c, e, g>0$. Let the areas of the triangles each formed by two sides and a diagonal be given by

$$m(ABC) \equiv x, \quad m(BCD) = y, \quad m(CDE) = z, \quad m(DEA) = u, \quad m(EAB) = v,$$

and we seek first $S=m(ABD)$. We assume that the vertices A, B, C, D, E indicate a counter-clockwise circuit of the pentagon. Hence the circuit of each triangle ABC, etc., is counter-clockwise—we adopt the usual convention that the area bounded by a counterclockwise circuit is positive. Thus, the convexity of $ABCDE$ implies that x, y, z, u, v are all positive.

It is easy to compute:

$$2x = ac \tag{1}$$

$$2y = e(b-a) + c(a-d) \tag{2}$$

$$2z = (d-b)(g-c) + (e-c)(b-f) \tag{3}$$

$$2u = dg - ef \tag{4}$$

$$2v = ag \tag{5}$$

$$2S = ae. \tag{6}$$

We eliminate c, e, g by substituting from (1), (5), (6) into (2), (3), (4). There results:

$$ay = bS - aS + ax - dx \tag{2'}$$

$$az = au - bv - dx + bS + fx \tag{3'}$$

$$au = dv - fS. \tag{4'}$$

Elimination of f between (3′) and (4′) gives:

$$azS + aux = auS - bvS - dxS + dxv + bS^2. \tag{7}$$

When we eliminate d between (2′) and (7), we note that b also goes out, giving:

$$S^2 + S\,(u+y-x-z-v) = ux - vx + vy. \tag{8}$$

Since the pentagon can be broken into triangles ABD, BCD and DEA, its area P is given by $P=S+y+u$. When we solve for P we then have

$$P = \tfrac{1}{2}(x + y + z + u + v) + \tfrac{1}{2}\sqrt{(x + y + z + u + v)^2 - 4(xy + yz + zu + uv + vx)};$$

therefore $S = m(ABD) = P - y - u$. Other dissections of the pentagon give directly

$$m(ACE) = P - x - z, \qquad m(BDE) = P - y - v,$$
$$m(BCE) = P - v - z, \qquad m(ACD) = P - x - u.$$

The positive sign is indicated before the radical in the expression for P since it is evident from a figure that if the pentagon is convex, $x+y+z+u+v$ does not cover the pentagon area twice.

While the area of the pentagon is uniquely determined by the areas x, y, z, u, v, the pentagon itself is not. Plainly, one transformation, or any sequence of several of the transformations

$$\mathrm{I}\begin{cases} y' = \alpha y \\ x' = x/\alpha, \end{cases} \qquad \mathrm{II}\begin{cases} y' = y + \beta x \\ x' = x, \end{cases} \qquad \mathrm{III}\begin{cases} y' = y \\ x' = x + \gamma y, \end{cases}$$

applied to all the vertices, changes the appearance of the pentagon, but leaves all the areas unchanged, as is easily seen from the matrix formula for the area of a triangle. (Here α, β, γ are arbitrary constants, the x's are abscissas and the y's are ordinates.)

For *any* set of five positive numbers x, y, z, u, v convex pentagons exist having the corresponding areas. In fact, we can determine the coordinates of the vertices: choose a, e such that $ae = 2S = 2P - 2y - 2u$; then c, g are determined from (1) and (6); with b arbitrary, positive, (2) gives d and finally (4) gives f. There are therefore many pentagons having the same x, y, z, u, v. However, if we call two pentagons *equivalent* if one can be obtained from the other by a sequence of transformations from I, II, III, then it is not difficult to show that all solutions of $a, b, \cdots, g$ in terms of $x, y, \cdots, v$ result in equivalent pentagons.

In the case of a *hexagon*, the areas of the six triangles, each formed by two sides and a diagonal, are *not* sufficient to determine the area of the hexagon or any of its other triangles.

Also solved by Michael Goldberg, Washington, D. C.; and the proposer.

Integral Distances

718. [January, 1969] *Proposed by A. H. Lumpkin, East Texas State University.*

In $R \times R$ with the usual metric, if G is an infinite subset of $R \times R$ such that for all x, y in G, $d(x, y)$ is an integer, then $G \subseteq l$ for some line l.

I. *Solution by Michael J. Martino, Temple University.*

We will show that it is impossible to construct G with the stated properties if $G \nsubseteq 1$, for some line 1 in $R \times R$.

It is sufficient to show that G cannot be constructed from lattice points along the positive x-axis if at least one point, p, does not lie on $y = 0$. By symmetry, the argument applies to the entire x-axis, and, since rotations and translations are distance preserving, to any line in $R\times R$.

1) Let $0=(0, 0)$, $x=(x, 0)$, and $p=(x, y)$ be elements of G that form a Pythagorean triangle in the first quadrant where x, y are integers. Then there is at most one other point in G, say $x_1=(x_1, 0)$, such that for all x, y in G, $d(x, y)$ is an integer.

2) If it were otherwise, there would be an infinite number of points $x_i=(x_i, 0)$ in G that would satisfy

$$d^2 = (x_i - x)^2 + y^2$$

in integers for a given y. This is not the case, hence G infinite implies $p=(x_i, 0)$ and the proof is complete.

II. *Comment by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

This is a known result due to Paul Erdös, *Integral distances*, Bull. Amer. Math. Soc., 51, 1945, p. 996. See also H. Hadwiger, H. Debrunner, and V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, New York, 1964, p. 5.

As shown in the latter reference, the result does not imply the existence of a number k_0 such that the conclusion always holds when the number k of points with exclusively integral distances is greater than k_0.

Additionally, there's no need for the "symbolic" language of the proposal. It could have been stated simply as: "If an infinite set of points in the plane is such that all of its points are at integral distances from each other, then all the points lie on a single line."

Solutions or references also submitted by William F. Fox, Moberly Junior College, Missouri; Michael Goldberg, Washington, D.C.; J. F. Leetch, Bowling Green State University, Ohio; Simeon Reich, Israel Institute of Technology, Haifa, Israel; and the proposer.

Comment on Problem 450

450. [May, 1961] *Proposed by Norman Anning, Sunnyvale, California.*

If the exponents m and n are positive integers, find the complete condition or conditions that x^m+x^n+1 shall have a polynomial factor other than itself and 1.

Comment by Andrzej Makowski, Warsaw, Poland.

The answer is given by the theorem of H. Trevberg (see his paper on the irreducibility of certain trinomials and quadrinomials in Math. Scand., 8, 1960, pages 65–70). The trinomial

$$f(x) = x^n + ee_1x^m + e_1$$

is irreducible whenever no root of $f(x)$ has the modulus 1. If $f(x)$ has roots with modulus 1, these roots can be collected to give a rational factor of $f(x)$. The other factor of $f(x)$ is then irreducible.

Comment on Problem 687

687. [March and November, 1968] *Proposed by Sidney H. L. King, Jacksonville University, Florida.*

Prove that if the perimeter of a quadrilateral $ABCD$ is cut into two portions of equal length by all straight lines passing through a fixed point O in it, the quadrilateral is a parallelogram.

Comment by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.

1. In the statement of the problem, "quadrilateral" should be replaced by "simple quadrilateral"; otherwise we could have Figure 1 as a solution.

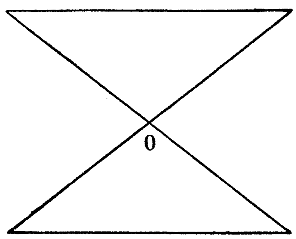

FIG. 1

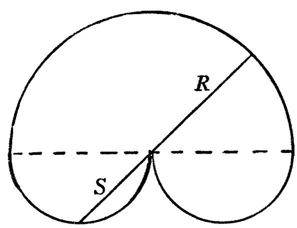

FIG. 2

2. The second solution given by myself is erroneous. The sophomoric error is in the equation $Rd\theta = Sd\theta$. This equation should have been

$$R^2 + \left(\frac{dR}{d\theta}\right)^2 = S^2 + \left(\frac{dS}{d\theta}\right)^2.$$

It now does not necessarily follow that $R = S$ to give a centrosymmetric figure. As a nice counterexample, consider Figure 2 made up from three semicircles. Every line through O bisects the perimeter. It would be of interest to find a noncentrosymmetric convex counterexample. However, if we restrict the figure to be a simple polygon, then Goldberg's solution implies that the polygon is centrosymmetric.

Comment on Problem 711

711. [November, 1968, and May, 1969] *Proposed by Thomas Shewczyk, University of Wisconsin at Waukesha.*

If the numbers $a_1, a_2, \cdots, a_n$ are positive, then show that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \geqq n^2.$$

Comment by Calvin T. Long, Washington State University.

The desired result is an immediate consequence of the well-known theorem of the arithmetic and geometric means. Thus, we have that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) \geqq \left(\prod_{i=1}^{n} a_i\right)^{1/n}\left(\prod_{i=1}^{n}\frac{1}{a_i}\right)^{1/n} = 1$$

and this yields the desired inequality. As a matter of fact, this is a special case of the following more general result. Let S_i and S_i^* be the ith elementary symmetric functions of the positive numbers $a_1, a_2, \cdots, a_n$ and $a_1^{-1}, a_2^{-1}, \cdots, a_n^{-1}$ respectively. Then we have

$$S_iS_i^* \geqq \binom{n}{i}^2, \qquad 1 \leqq i \leqq n,$$

and the present inequality is simply the case $i=1$. The more general result follows as above from the not-so-well-known inequality

$$\left[\frac{S_n}{\binom{n}{n}}\right]^{1/n} \leqq \left[\frac{S_{n-1}}{\binom{n}{n-1}}\right]^{1/(n-1)} \leqq \cdots \leqq \left[\frac{S_1}{\binom{n}{1}}\right]$$

which can be found in *Inequalities*, by Hardy, Littlewood, and Polya.

Comment on Q424

Q424. [January, 1968] Prove that the legs of a right triangle cannot have their lengths equal to twin primes.

[*Submitted by John H. Tiner*]

Comment by Charles W. Trigg, San Diego, California.

Of course, the legs of a *right triangle can* have lengths equal to twin primes, but the legs of a *Pythagorean triangle cannot* both be odd numbers. Indeed, it is standard practice to represent the legs of a primitive triangle as $a=m^2-n^2$, $b=2mn$, where m and n are relatively prime and one is odd and the other even. Thus a is always odd and b is even. In the Pythagorean triangles derived from the primitive ones, both legs may be even, but never are both odd.

The method used in **A424** can be used to arrive at the same conclusion about two odd legs.

Comment on Q426

Q426. [March, 1968] Without using calculus, determine the least value of the function $f(x)=(x+a+b)(x+a-b)(x-a+b)(x-a-b)$, where a and b are real constants.

[*Submitted by Roger B. Eggleton*]

Comment by S. Spital, California State College at Hayward.

A solution is obtained by multiplying out and completing the square:

$$\begin{aligned} f(x) &= [x^2-(a+b)^2][x^2-(a-b)^2] \\ &= [x^2-(a^2+b^2)]^2 - 4a^2b^2 \geqq -4a^2b^2. \end{aligned}$$

Comment on Q442

Q442. [November, 1968] E. T. Bell ("Men of Mathematics") relates an amusing story of Descartes assigning his pupil, Catherine the Great, the famous Appollonian problem of constructing a circle tangent to three given circles. To vent his hidden contempt for her scholarly pretensions, he neglected to warn the poor girl that synthetic geometry should be used. She used his new analytic geometry and, supposedly, was led into a trap, the required solution of three simultaneous quadratics. On the contrary, show how an easy triumph was possible.

Comment by Donald R. Morrison, Sandia Corporation, Albuquerque, New Mexico.

The solution to Problem **Q442** begins, "Suppose Catherine did not share with Descartes a prejudice against negative radii." The solution involves the expression

$$\begin{bmatrix} h_1 - h_2 & k_1 - k_2 & r_1 - r_2 \\ h_2 - h_3 & k_2 - k_3 & r_2 - r_3 \\ h_3 - h_1 & k_3 - k_1 & r_3 - r_1 \end{bmatrix}^{-1}.$$

To accept this solution, Catherine would have had to reject, also, Descartes' prejudice against inverses of singular matrices. The matrix is obviously singular, since the sum of its row vectors is the zero vector.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q458. The functions defined by $f(x)=0$ and $f(x)=x^n$ have the property that for any integer m, $[f(x)]^m \equiv f(x^m)$. Prove that there exists a real valued function with this property which is discontinuous at each point x in $(-\infty, \infty)$.

[*Submitted by Erwin Just*]

Q459. Find the fallacy in the following:

(a) $$x^2 = (x)(x) = x + x + \cdots + x$$

(a total of x addends)

(b) $$\frac{d(x^2)}{dx} = \frac{d(x + x + \cdots + x)}{dx}$$

(a total of x addends)

(c) $$2x = 1 + 1 + \cdots + 1$$

(a total of x addends)

(d) $2x = x$

(e) $2 = 1.$

[*Submitted by Steven R. Conrad*]

Q460. Find all points of intersection of the curves given in polar form by the equations $r=\cos\theta/2$ and $r=\sin\theta/2$.

[*Submitted by John Beidler*]

Q461. If (a, b, c) is a Pythagorean triple and $\alpha=\text{arc}\sin a/c=\text{arc}\cos b/c$, then $2\alpha=\text{arc}\sin e/g=\text{arc}\cos f/g$ such that $(|e|, |f|, g)$ is also a Pythagorean triple.

[*Submitted by Ann B. Brigham and Charles K. Brown, III*]

Q462. Does there exist a real function f such that $f(x+p)=f(x)$ for every $p>0$ if f is not a constant?

[*Submitted by M. L. Bittinger*]

(Answers on page 203)

ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1969 recipients of these awards, selected by a committee consisting of Ivan Niven, Chairman; Edwin Hewitt, and D. E. Richmond, were announced by President Young at the Business Meeting of the Association on August 26, 1969, at the University of Oregon. The recipients of the Ford Awards for articles published in 1968 were the following:

Harley Flanders, A Proof of Minkowski's Inequality for Convex Curves, MONTHLY, 75 (1968) 581–593.

G. E. Forsythe, What to do Till the Computer Scientist Comes, MONTHLY, 75 (1968) 454–462.

M. F. Neuts, Are Many 1-1 Functions on the Positive Integers Onto?, this MAGAZINE, 41 (1968) 103–109.

Pierre Samuel, Unique Factorization, MONTHLY, 75 (1968) 945–952.

Hassler Whitney, The Mathematics of Physical Quantities, I and II, MONTHLY, 75 (1968) 115–138 and 227–256.

Albert Wilansky, Spectral Decomposition of Matrices for High School Students, this MAGAZINE, 41 (1968) 51–59.

HENRY L. ALDER, *Secretary*

ANSWERS

A458. Define f such that $f(x)=0$ when x is algebraic and $f(x)=1$ when x is transcendental. f is discontinuous at each point, and since k and k^m are both algebraic or both transcendental, f has the desired property.

A459. In the first step, the definition of multiplication as repeated addition is valid only for integers. Hence, the function is not continuous and differentiation is meaningless for this definition.

A460. Since in the equation $r=\cos\theta/2$, θ may be replaced by $-\theta$ and the equation remains the same, the curve represented by the equation is symmetric with respect to the x axis. Next, r may be replaced by $-r$ and θ by $3\pi-\theta$ to yield a new equation for the same curve. The result is

$$-r = \cos(3\pi - \theta/2) = \cos 3\pi/2 \cos\theta/2 + \sin 3\pi/2 \sin\theta/2 = -\sin\theta/2$$

or $r=\sin\theta/2$. Hence the two curves are identical.

A 461.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2ab}{c^2} = \frac{2ab}{a^2+b^2} = e/g$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{b^2-a^2}{c^2} = \frac{b^2-a^2}{a^2+b^2} = b/g.$$

But $2ab$, a^2+b^2, and a^2-b^2 are generators of Pythagorean triples and the result follows.

A462. No. Suppose for contradiction that there exists such an f. Then there is an x_1 and x_2, $x_1\neq x_2$, such that $f(x_1)\neq f(x_2)$. Assume without loss of generality that $x_1<x_2$. Then there exists a $p'>0$ such that $x_1+p'=x_2$ and $f(x_1+p')=f(y)$ $=f(x)$, a contradiction.

(Quickies on page 225)

AN ALTERNATIVE TO THE GRAM-SCHMIDT PROCESS

JOHN H. STAIB, Drexel Institute of Technology

I dislike presenting the Gram-Schmidt process: it lacks elegance (in my opinion); it is not easy for the student to remember; and it is arithmetically so cumbersome that it makes testing of the student most difficult. Happily, while preparing for a lecture on the Gram-Schmidt process, I stumbled upon an alternative: a matrix method. If it is not original, it is at least not widely known.

We shall need some notation: E is an n-dimensional Euclidean space over the reals; $\mathcal{B}$ is an n by 1 column of independent vectors from E (thus, $\mathcal{B}$ represents an ordered basis for E); and $X_{\mathcal{B}}$ is an n by 1 column whose entries are the $\mathcal{B}$-coordinates of a vector X.

(d) $$2x = x$$

(e) $$2 = 1.$$

[*Submitted by Steven R. Conrad*]

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